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An analysis of convergence of hybridized approximation techniques for the analytical solution of partial differential equations

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ABSTRACT: In this paper, we use the hybridization of two well-known semi-analytical methods to obtain the numerical solutions for some special linear and nonlinear partial differential equations that are predominant in most physical science disciplines. The first among these methods is the coupling of Laplace transform and Adomian decomposition method (LADM) while the second method is standard Homotopy perturbation method with Laplace transform method (HPTM). The accuracy and dependability of these proposed techniques is confirmed by applying them to solve linear and nonlinear Kleidon-Gordon equations, linear transverse equation of a vibrating beam, homogeneous and inhomogeneous nonlinear PDEs, advection equation, diffusion-convection and Korteweg-DeVries equation. Thereafter, comparison between the solutions obtained by the methods is presented in tables for convergence analysis. Consequently, the findings from our study showed the two methods can be effective alternative approaches for obtaining solutions to linear and nonlinear PDEs and higher-order initial value problems.

KEYWORDS: Semi-analytical techniques, NLPDEs, LADM, HPSTM, Convergence Analysis

INTRODUCTION

The nonlinear partial differential equation (PDE) is a challenging and critical area of mathematics that plays a crucial role in understanding a wide range of natural phenomena and physical processes. Nonlinear PDEs describe complex interactions and behaviour that occur across many scientific disciplines, such as physics, engineering, biology, and environmental science, in contrast to their linear counterparts. The study of nonlinear PDEs has become a central focus of research in applied mathematics and theoretical physics for unravelling intricate patterns, wave phenomena, and dynamic systems [1-3]. A nonlinear PDE can be found in many contexts, including fluid dynamics, heat transfer, and quantum mechanics. Often, their inherent complexity requires sophisticated mathematical methods to analyse and solve. To unravel the underlying dynamics encoded in these nonlinear equations, researchers and practitioners use a variety of methods, including numerical simulations and analytical approaches. Understanding and solving nonlinear PDEs contribute significantly to advancing our comprehension of intricate physical processes and phenomena [4-6]

Nonlinear PDEs are prevalent in many scientific disciplines, which highlights their significance. Nonlinear PDEs describe phase transitions and nonlinear diffusion processes in material science, and they model the complex interactions between fluid flow, turbulence, and wave propagation in fluid dynamics. Nonlinear PDEs are extremely useful in mathematical biology, where they help explain the dynamics of disease propagation, pattern development, and population increase. Several methods ranging from analytical, numerical, and semi-analytical have been used to solve these problems to a great effect. Some of these innovative mathematical techniques include Adomian decomposition method (ADM), Homotopy perturbation method (HPM), Homotopy analysis method (HAM), Variation iteration method (VIM), Differential transform method (DTM), Differential quadrature method (DQM), Abkari-Ganji method (AGM), Temimi-Ansari method (TAM), Daftardar-Jafari method (DJM) and others [3-5]. Similarly, the combination of two or more of these semi-inverse techniques have been considered especially to nonlinear problems to evade the nonlinearity which is often characterized by difficulty in giving rise to analytical or approximate solution. [7-9]

The coupling of the Laplace transforms method and the standard Adomian decomposition procedure have been given extensive attention among academics in literature. Numerous benefits come from studying Laplace Adomian decomposition method (LADM). The first benefit of using LADM is that it solves fractional differential equations, both linear and nonlinear, rapidly, and efficiently by reducing the equation's structure into a set of recursive schemes. LADM is a dependable technique since it yields a sequence of Adomian components that accurately solves the problem. Furthermore, LADM is a semi-analytical technique that creates hybrid methods that can handle different kinds of differential equations by combining the Adomian decomposition method with different transformation methods like Laplace transform, Sumudu transform, Elzaki transform, and Aboodh transform. Lastly, LADM is also helpful to neuroscientists and clinical research in medicine since it enables the visualization of solitons and numerical simulations with an impressively small error measure. Due to the ease of implementation of this method it has been effectively implemented to solve diverse problems. Roohani et al. [10] used Laplace-Adomian decomposition method to look at the consequences of slip effects on magnetohydrodynamics viscous flow through a permeable stretching sheet. Utilizing the pade approximation, the obtained analytical solution was improved upon in the convergence domain. Ebiwareme et al. [11-16] have employed the Laplace Adomian decomposition method to tackle the food chain ecoepidemic model, SIR infectious disease model, dynamics of the Hepatitis E virus, modelling the flow of atmospheric CO2 in the atmosphere, crime deterrence model and magnetohydrodynamic flow of incompressible fluid between parallel plates.

Similarly, the Homotopy Perturbation Transform Method (HPTM) is a potent mathematical technique that has gained popularity recently because of its effectiveness in resolving a variety of challenging nonlinear problems. The Homotopy Perturbation Method (HPTM), which was developed as an extension of the HPM, combines the advantages of transforms and perturbation methods to produce precise and effective solutions to nonlinear differential equations. This approach has shown to be especially beneficial when addressing problems in a variety of technical and scientific fields. The primary advantage of HPTM is its capacity to produce extremely precise analytical solutions for nonlinear problems. Using modifications and perturbation techniques, HPTM makes it possible to generate closed-form solutions, providing insights into the underlying behaviour of complex systems. Second, physics, engineering, and applied mathematics are fields where nonlinear problems are frequently encountered. This is where HPTM thrives. Its adaptability to nonlinearity of different orders makes it a useful tool for researchers and practitioners studying real-world occurrences. Its favourable convergence features also guarantee the dependability of its solutions. Its underlying principles are relatively simple, making it accessible to researchers and engineers with varying levels of mathematical expertise. This makes it a preferred choice when compared to other analytical and numerical methods, especially in situations where convergence might be challenging for traditional techniques. HPTM is appealing in a variety of disciplines because to its intuitiveness, which makes it easier to apply to a wide range of situations. It also strikes a balance between computational economy and analytical elegance. When precision and computing cost are considered, it is a preferred option since it offers correct solutions with less computational work than some numerical methods. [17-31]. Ebiwareme et al. [32-34] have employed to solve physical models comprising complicated PDEs, equations governing Jeffrey-Hamel flow and analysis of heat and Mass transfer of convective fluid passing through a vertical porous plate under the impression of chemical reaction with inclined magnetic field.

The current study examines the convergence analysis of a few specific nonlinear partial differential equations by combining the conventional Laplace transform approach with the Adomian decomposition method and the Homotopy perturbation method. These hybrid semi-analytical techniques are successfully applied to numerical examples in order to approximate the analytical solutions for the following: linear and nonlinear Kleidon-Gordon equations; linear transverse equation of a vibrating beam; homogeneous and inhomogeneous nonlinear PDEs; advection equation; diffusion-convection; and Korteweg-DeVries equation. While Adomian and He's polynomials make dealing with nonlinear terms straightforward noise terms are found in all nonlinear problems that are examined. The structure of the study is as follows: The foundations of the problem-solving methods are covered in sections two and three. Section four solves illustrative cases of both linear and nonlinear PDEs, leading to fast convergent solutions. In Section Six, a conclusion to the study is presented, while in Section Five, the findings of the obtained analytical solutions are presented in tables and 3D graphics. The outcome corresponds with previous research, giving a precise, elegant, and computationally reliable approach.

2 Laplace Adomian Decomposition method (LADM)

In this subsection, fundamentals of the fusion of Laplace transformation and Adomian decomposition method (LADM) is presented. Consider a functional differential equation of the form.

$$L[u(x)] + R[u(x)] + N[u(x)] = g(x) \quad (1)$$

Subject to the initial condition

$$u(x,0) = f(x), \ \frac{\partial u(x,0)}{\partial t} = h(x) \quad (2)$$

Rearranging the above, we obtain the following relation for $L[u(x)]$

$$L[u(x)] = g(x) - R[u(x)] - N[u(x)] \quad (3)$$

Applying Laplace transform on both sides of Eq. (1), supposing the highest differential operator is of order two and using the differentiation property, we get.

$$s^2\mathcal{L}\{u(x)\} - sh(x) - f(x) = \mathcal{L}\{g(x)\} - \mathcal{L}\{Ru(x)\} - \mathcal{L}\{Nu(x)\}$$

$$s^2\mathcal{L}\{u(x)\} = sh(x) + f(x) + \mathcal{L}\{g(x)\} - \mathcal{L}\{Ru(x)\} - \mathcal{L}\{Nu(x)\}$$

$$\mathcal{L}\{u(x)\} = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}\mathcal{L}\{g(x)\} - \frac{1}{s^2}\mathcal{L}\{Ru(x)\} - \frac{1}{s^2}\mathcal{L}\{Nu(x)\} \quad (4)$$

Next, we apply the inverse transform on both sides of Eq. (4), we obtain.

$$u(x) = \phi(x) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\{Ru(x)\} - \frac{1}{s^2}\mathcal{L}\{Nu(x)\}\right] \quad (5)$$

where $\phi(x)$ is the term arising from the first three terms on the right-hand side of Eq. (5)

Next, we assume the solution as decomposing series in the form.

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (6)$$

Similarly, the nonlinear terms are written in terms of the Adomian polynomials.

$$Nu(x) = \sum_{n=0}^{\infty} A_n \quad (7)$$

where the $A_n^{\prime s}$ represents the Adomian polynomials defined in the form

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}\lambda^i y_i\right)\right]_{i=0}, n = 0,1,2,3 \tag{8}$$

Plugging Eqs. (8) and (9) into Eq. (5), we obtain

$$\sum_{n=0}^{\infty} u_n(x) = \phi(x) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\{R\sum_{n=0}^{\infty} u_n(x)\} - \frac{1}{s^2}\mathcal{L}\{N\sum_{n=0}^{\infty} A_n\}\right] \tag{9}$$

Matching both sides of Eq. (9), we obtain an iterative algorithm in the form.

$$\begin{aligned}
u_0(x) &= \phi(x) \\
u_1(x) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left\{R\sum_{n=0}^{\infty} u_0(x)\right\} - \frac{1}{s^2}\mathcal{L}\left\{N\sum_{n=0}^{\infty} A_0\right\}\right] \\
u_2(x) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\{R\sum_{n=0}^{\infty} u_1(x)\} - \frac{1}{s^2}\mathcal{L}\{N\sum_{n=0}^{\infty} A_1\}\right] \\
u_3(x) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left\{R\sum_{n=0}^{\infty} u_2(x)\right\} - \frac{1}{s^2}\mathcal{L}\left\{N\sum_{n=0}^{\infty} A_2\right\}\right] \\
&\vdots \\
u_{n+1}(x) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left\{R\sum_{n=0}^{\infty} u_n(x)\right\} - \frac{1}{s^2}\mathcal{L}\left\{N\sum_{n=0}^{\infty} A_n\right\}\right]
\end{aligned} \tag{10}$$

Then the solution of the differential equation is obtained as the sum of decomposed series in the form

$$u(x) \approx u_0(x) + u_1(x) + u_2(x) + \cdots \tag{11}$$

3 Homotopy perturbation transform method (HPTM)

In this subsection, we give the basics of coupling homotopy perturbation method with the Laplace transformation, called homotopy perturbation transform method (HPTM) for solving the linear and nonlinear PDEs. Consider the functional second order differential equations of the form.

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{12}$$

$$u(x,0) = \phi(x), u_t(x,0) = f(x) \tag{13}$$

where D denote a second order linear differential operator, R represents linear differential operator of order less than D, N is a nonlinear differential operator and $g(x)$ is called the source term.

Taking the Laplace transform of both sides of Eq. (22), we have the expression.

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \tag{14}$$

$$s^2 L[u(x,t)] - su(x,0) - u_t(x,0) + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \tag{15}$$

Invoking the initial condition into Eq. (15), we obtain

$$L[u(x,t)] = \frac{\xi(x)}{s} + \frac{f(x)}{s} + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Ru(x,t)] - \frac{1}{s^2}L[Nu(x,t)] \tag{16}$$

Taking the inverse Laplace transform of both sides of Eq. (26), we have.

$$u(x,t) = \psi(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}L[Ru(x,t)] + \frac{1}{s^2}L[Nu(x,t)]\right] \tag{17}$$

where $\psi(x,t)$ is the term that originates from the integration of the source term subject to the prescribed initial conditions

Next, we apply the HPM for the solution as well as the nonlinear terms as follows.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{18}$$

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{19}$$

where $H_n(u)$ are the so-called He's polynomials defined by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{k=0}^{n} p^n u_n\right)\right)_{p=0}, p = 0,1,2 \tag{20}$$

Substituting Eqs. (18) and (19) into Eq. (17), we have the equivalent expression as

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \psi(x,t) - p\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n\big(u(x,t)\big)\right]\right]\right) \tag{21}$$

Comparing the coefficients of like powers of p, we obtain the following approximations given as

$$\begin{aligned}
p^0&: u_0(x,t) = \psi(x,t) \\
p^1&: u_1(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}L\left[Ru_0(x,t) + H_0\big(u(x,t)\big)\right]\right] \\
p^2&: u_2(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}L\left[Ru_1(x,t) + H_1\big(u(x,t)\big)\right]\right]
\end{aligned} \tag{22}$$

$$p^3: u_3(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}L\left[Ru_2(x,t) + H_2\big(u(x,t)\big)\right]\right]$$
$$p^4: u_4(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}L\left[Ru_3(x,t) + H_3\big(u(x,t)\big)\right]\right]$$
$$\vdots$$

Using Eq. (22), the approximate solution of the given problem takes the form.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{23}$$

4 Analysis of Illustrative Examples using HPTM

In this section, we apply the proposed hybrid techniques to solve eight linear and nonlinear partial differential equations namely: linear and nonlinear Kleidon-Gordon equations, linear transverse equation of a vibrating beam, homogeneous and inhomogeneous nonlinear PDEs, advection equation, diffusion-convection and KDV equation respectively. The iterative approximations from the recursive algorithms are computed using Mathematica version 13.

Example 1 Consider the linear Klein-Gordon equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u \tag{24}$$

Subject to the initial conditions, $u(x,0) = 1 + \sin x, u_t(x,0) = 0$

Taking the Laplace transform of both sides and invoking the initial conditions, we have.

$$u(x,s) = \frac{1+\sin x}{s} + \frac{1}{s^2}L\left(u + \frac{\partial^2 u}{\partial x^2}\right) \tag{25}$$

Taking the inverse Laplace transformation of both sides of the above, we have

$$u(x,t) = 1 + \sin x + \frac{1}{s^2}L\left(u + \frac{\partial^2 u}{\partial x^2}\right) \tag{26}$$

Next, apply the Homotopy perturbation method on the above equation, we have.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{27}$$

Plugging the above into the given equation we obtain the expression of the form.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + \sin x + L^{-1}\left[\frac{1}{s^2} L\left(\sum_{n=0}^{\infty} p^n u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right] \tag{28}$$

Comparing the coefficients of like powers of p, we have the approximate solutions.

$$\begin{aligned}
&p^0: u_0(x,t) = 1 + \sin x \\
&p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L\left(u_0(x,t) + \frac{\partial^2}{\partial x^2} u_0(x,t)\right)\right] = \frac{t^2}{2!} \\
&p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L\left(u_1(x,t) + \frac{\partial^2}{\partial x^2} u_1(x,t)\right)\right] = \frac{t^4}{4!} \\
&p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s^2} L\left(u_2(x,t) + \frac{\partial^2}{\partial x^2} u_2(x,t)\right)\right] = \frac{t^6}{6!} \\
&\vdots
\end{aligned} \tag{29}$$

Using the relation, $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$, the approximate solution become

$$u(x,t) = 1 + \sin x + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots \tag{30}$$

The closed form solution of the above in series form is gives the expression.

$$u(x,t) = \sin x + \cosh t \tag{31}$$

Example 2. Consider the following homogenous nonlinear PDE.

$$\frac{\partial u}{\partial t} - u - u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 = 0 \tag{32}$$

With the initial condition, $u(x,0) = \sqrt{x}$

Taking the Laplace transform of both sides with respect to t subject to the initial condition, we have.

$$u(x,s) = \frac{\sqrt{x}}{s} + \frac{1}{s} L\left[u + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \tag{33}$$

Taking the inverse Laplace transform of both sides of the above gives.

$$u(x,t) = \sqrt{x} + L^{-1}\left[\frac{1}{s}L\left[u + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]\right] \tag{34}$$

Next, we apply the Homotopy perturbation method by writing the solution term as a decomposition series.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Plugging the above expression into the above equation gives the form.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sqrt{x} + p\left(L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} p^n u_n(x,t)] + \frac{1}{s}L[\sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \tag{35}$$

where H_n are the He's polynomials that represents the nonlinear terms. The first few components of these He's polynomials are given as

$$\begin{gathered} H_0(u) = u_0\frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_0}{\partial x}\right)^2 = 0 \\ H_1(u) = u_0\frac{\partial^2 u_1}{\partial x^2} + u_1\frac{\partial^2 u_0}{\partial x^2} + 2\frac{\partial u_0}{\partial x}.\frac{\partial u_1}{\partial x} = 0 \\ H_2(u) = u_0\frac{\partial^2 u_2}{\partial x^2} + u_1\frac{\partial^2 u_1}{\partial x^2} + u_2\frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}.\frac{\partial u_2}{\partial x} = 0 \\ \vdots \end{gathered} \tag{36}$$

Comparing the coefficients of like powers of p, we have the iterative approximations as

$$\begin{aligned} &p^0: u_0(x,t) = \sqrt{x} \\ &p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}(L[u_0(x,t)] + L[H_0(u)])\right] = \sqrt{x}t \\ &p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}(L[u_1(x,t)] + L[H_1(u)])\right] = \sqrt{x}\frac{t^2}{2!} \\ &p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s}(L[u_2(x,t)] + L[H_2(u)])\right] = \sqrt{x}\frac{t^3}{3!} \\ &\vdots \end{aligned} \tag{37}$$

And so, the approximate series solution of the problem is given as

$$u(x,t) = \sqrt{x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) = \sqrt{x}e^t \tag{38}$$

Example 3. Consider the following Advection problem.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2 \quad (39)$$

with the initial condition, $u(x,0) = 0$

Taking the Laplace transform of both sides subject to the initial condition, we have.

$$u(x,s) = \frac{2}{s^3} + \frac{x}{s^2} + \frac{6}{s^5} + \frac{2x}{s^4} - \frac{1}{s}L\left[u\frac{\partial u}{\partial x}\right] \quad (40)$$

Taking the inverse Laplace transform of both sides gives.

$$u(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - L^{-1}\left[\frac{1}{s}L\left[u\frac{\partial u}{\partial x}\right]\right] \quad (41)$$

Now, we apply the Homotopy perturbation method, we have.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Plugging the above into the given problem gives the form.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - p\left(L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} p^n u_n(x,t)]\right]\right) \quad (42)$$

where H_n are the so-called He's polynomials that represents the nonlinear terms. The first few components are given as

$$H_0(u) = u_0 u_{0x}$$
$$H_1(u) = u_0 u_{1x} + u_1 u_{0x}$$
$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \quad (43)$$
$$H_3(u) = u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x}$$
$$\vdots$$

Comparing the coefficients of the like powers of p, we have the expressions for the iterative solutions given as

$$p^0 : u_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[H_0(u)]\right] = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} + \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{98} \quad (44)$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[H_1(u)]\right]$$
$$= \frac{5t^{12}}{8064} + \frac{2xt^{11}}{2079} + \frac{2783t^{10}}{302400} + \frac{38xt^9}{2835} + \frac{143t^8}{2880} + \frac{22xt^7}{315} + \frac{7t^6}{12} + \frac{2xt^5}{15}$$

Cancelling out the noise terms in the components, $u_0(x,t)$ and $u_1(x,t)$. The exact solution of the problem become.

$$u(x,t) = t^2 + xt \quad (45)$$

Example 4. Consider the following diffusion-convection problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad (46)$$

with the initial conditions, $u(x,0) = x + e^{-x}$

Taking the Laplace transform of both sides of the given equation and invoking the initial conditions gives the form.

$$u(x,s) = \frac{x+e^{-x}}{s} + \frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u\right] \quad (47)$$

Taking the inverse Laplace transform of both sides, we obtain.

$$u(x,t) = x + e^{-x} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u\right]\right] \quad (48)$$

Next, we apply the Homotopy perturbation method, we get.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Plugging the above expression into the inverse equation, we have the form.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + e^{-x} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \quad (49)$$

Comparing the coefficients of like powers of p, we have

$$p^0: u_0(x,t) = x + e^{-x}$$
$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_0}{\partial x^2} - u_0(x,t)\right]\right] = -xt$$
$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_1}{\partial x^2} - u_1(x,t)\right]\right] = x\frac{t^2}{2!} \tag{50}$$
$$p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_2}{\partial x^2} - u_2(x,t)\right]\right] = -x\frac{t^3}{3!}$$
$$\vdots$$

And so on, therefore the series solution is given in the form.

$$u(x,t) = e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

which converges rapidly to the exact solution given as

$$u(x,t) = e^{-x} + xe^{-t} \tag{51}$$

Example 5. Consider the inhomogeneous nonlinear Klein-Gordon equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x\cos t + x^2\cos^2 t \tag{52}$$

with the initial condition, $u(x,0) = x, u_t(x,0) = 0$

Taking the Laplace transform of both sides of the given equation subject to the initial condition and rearranging gives.

$$u(x,s) = \frac{x}{s} - \frac{x}{s(1+s^2)} + \frac{(2+s^2)x^2}{s^3(4+s^2)} + \frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u^2\right] \tag{53}$$

Taking the inverse Laplace transform gives the expression.

$$u(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \frac{1}{4}x^2t^2 + \frac{x^2}{8} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]\right] \tag{54}$$

Next, we apply the homotopy perturbation method, we have.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Plugging into the given equation gives the expression in terms of the perturbation parameter.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \frac{1}{4}x^2t^2 + \frac{x^2}{8} + p\left(L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \quad (55)$$

where $H_n(u)$ represent the so-called He's polynomials. The first few of them are defined as

$$\begin{gathered} H_0(u) = u_0^2 \\ H_1(u) = 2u_0u_1 \\ H_2(u) = 2u_0u_2 + u_1^2 \\ H_1(u) = 2u_0u_3 + 2u_1u_2 \end{gathered} \quad (56)$$

Comparing the coefficients of like powers of p, we have the iterative approximations given as

$$p^0: u_0(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \left(\frac{xt}{2}\right)^2 + \frac{x^2}{8}$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_0}{\partial x^2} - H_0(u)\right]\right] = \frac{1}{8}x^2\cos t - \left(\frac{xt}{2}\right)^2 - \frac{x^2}{8} + \frac{1}{64}\cos 2t + \cdots \quad (57)$$

Cancelling out the noise terms in the $u_0(x,t)$ and $u_1(x,t)$, then the remaining terms of $u_0(x,t)$ which satisfies the given problem gives the solution as

$$u(x,t) = x\cos t \quad (58)$$

Example 6. Consider the following inhomogeneous nonlinear PDE given as

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 2x + t^4 \quad (59)$$

with the initial conditions, $u(x,0) = 0, u_t(x,0) = 2$

Taking the Laplace transform of both sides of the given problem subject to the initial condition and rearranging gives the resulting expression.

$$u(x,s) = \frac{2}{s^2} + \frac{2x}{s^3} + \frac{24}{s^7} - \frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \quad (60)$$

Taking the inverse Laplace transform of both sides, we have.

$$u(x,t) = 2t + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]\right] \quad (61)$$

Next, we apply the Homotopy perturbation method, we obtain.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Substitution into the above expression gives the expression.

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 2t + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \quad (62)$$

The first few terms of the He's polynomials denoted $H_n(u)$ are given by the expressions.

$$H_0(u) = \left(\frac{\partial u_0}{\partial x}\right)^2 = t^4$$

$$H_1(u) = 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} = 0$$

$$H_2(u) = \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} = 0 \quad (63)$$

$$H_3(u) = 2\frac{\partial u_0}{\partial x}\frac{\partial u_3}{\partial x} + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} = 0$$

Comparing the coefficients of like powers of p, the iterative approximations become

$$p^0: u_0(x,t) = 2t + xt^2 + \frac{t^6}{30}$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u_0}{\partial x^2} + H_0(u)\right]\right] = -\frac{t^6}{30} \quad (64)$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u_1}{\partial x^2} + H_1(u)\right]\right] = 0$$

$$\vdots$$

$$p^k: u_k(x,t) = 0, k \geq 2$$

Therefore, the exact solution of the problem is given as follows.

$$u(x,t) = 2t + xt^2 \tag{65}$$

5 Implementation of illustrative Examples using LADM

Example 1 Consider the linear Klein-Gordon equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u \tag{66}$$

Subject to the initial conditions, $u(x,0) = 1 + \sin x, u_t(x,0) = 0$

Taking the Laplace transform of both sides and invoking the initial conditions, we have.

$$u(x,s) = \frac{1+\sin x}{s} + \frac{1}{s^2}L\left(u + \frac{\partial^2 u}{\partial x^2}\right) \tag{67}$$

Taking the inverse Laplace transformation of both sides of the above, we have

$$u(x,t) = 1 + \sin x + \frac{1}{s^2}L\left(u + \frac{\partial^2 u}{\partial x^2}\right) \tag{68}$$

Next, writing the unknown as a decomposition series of the form, we have.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Plugging the above into the given equation we obtain the expression of the form.

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + \sin x + L^{-1}\left[\frac{1}{s^2}L\left(\sum_{n=0}^{\infty} u_n(x,t) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n(x,t)\right)\right] \tag{69}$$

Comparing the coefficients of like powers of both sides, we have the approximate solutions as

$$\begin{aligned} u_0(x,t) &= 1+\sin x \\ u_1(x,t) &= L^{-1}\left[\frac{1}{s^2}L\left(u_0(x,t)+\frac{\partial^2}{\partial x^2}u_0(x,t)\right)\right]=\frac{t^2}{2!} \\ u_2(x,t) &= L^{-1}\left[\frac{1}{s^2}L\left(u_1(x,t)+\frac{\partial^2}{\partial x^2}u_1(x,t)\right)\right]=\frac{t^4}{4!} \\ u_3(x,t) &= L^{-1}\left[\frac{1}{s^2}L\left(u_2(x,t)+\frac{\partial^2}{\partial x^2}u_2(x,t)\right)\right]=\frac{t^6}{6!} \\ &\vdots \end{aligned} \tag{70}$$

Using the relation, $u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots$, the approximate solution become

$$u(x,t)=1+\sin x+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\cdots$$

The closed form solution of the above in series form is gives the expression.

$$u(x,t)=\sin x+\cosh t \tag{71}$$

Example 2. Consider the following homogenous nonlinear PDE.

$$\frac{\partial u}{\partial t}-u-u\frac{\partial^2 u}{\partial x^2}-\left(\frac{\partial u}{\partial x}\right)^2=0 \tag{72}$$

with the initial condition, $u(x,0)=\sqrt{x}$

Taking the Laplace transform of both sides with respect to t subject to the initial condition, we have.

$$u(x,s)=\frac{\sqrt{x}}{s}+\frac{1}{s}L\left[u+u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{73}$$

Taking the inverse Laplace transform of both sides and substituting the initial condition gives.

$$u(x,t)=\sqrt{x}+L^{-1}\left[\frac{1}{s}L\left[u+u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]\right] \tag{74}$$

By the standard Adomian procedure, we write the solution term as a decomposition series.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Plugging the above expression into the above equation gives the form.

$$\sum_{n=0}^{\infty} u_n(x,t) = \sqrt{x} + \left(L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} u_n(x,t)\right] + \frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(u)\right]\right]\right) \quad (75)$$

where A_n are the Adomian polynomials that represents the nonlinear terms. The first few components of these Adomian polynomials are given as

$$A_0(u) = u_0\frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_0}{\partial x}\right)^2 = 0$$
$$A_1(u) = u_0\frac{\partial^2 u_1}{\partial x^2} + u_1\frac{\partial^2 u_0}{\partial x^2} + 2\frac{\partial u_0}{\partial x}.\frac{\partial u_1}{\partial x} = 0 \quad (76)$$
$$A_2(u) = u_0\frac{\partial^2 u_2}{\partial x^2} + u_1\frac{\partial^2 u_1}{\partial x^2} + u_2\frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}.\frac{\partial u_2}{\partial x} = 0$$
$$\vdots$$

Comparing the coefficients of both sides, we have the iterative approximations as

$$u_0(x,t) = \sqrt{x}$$
$$u_1(x,t) = L^{-1}\left[\frac{1}{s}(L[u_0(x,t)] + L[A_0(u)])\right] = \sqrt{x}t$$
$$u_2(x,t) = L^{-1}\left[\frac{1}{s}(L[u_1(x,t)] + L[A_1(u)])\right] = \sqrt{x}\frac{t^2}{2!} \quad (77)$$
$$u_3(x,t) = L^{-1}\left[\frac{1}{s}(L[u_2(x,t)] + L[A_2(u)])\right] = \sqrt{x}\frac{t^3}{3!}$$
$$\vdots$$

And so, the approximate series solution of the problem is given as

$$u(x,t) = \sqrt{x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) = \sqrt{x}e^t \quad (78)$$

Example 3. Consider the following Advection problem.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2 \tag{79}$$

with the initial condition, $u(x, 0) = 0$

Taking the Laplace transform of both sides subject to the initial condition, we have.

$$u(x, s) = \frac{2}{s^3} + \frac{x}{s^2} + \frac{6}{s^5} + \frac{2x}{s^4} - \frac{1}{s}L\left[u\frac{\partial u}{\partial x}\right] \tag{80}$$

Taking the inverse Laplace transform of both sides gives.

$$u(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - L^{-1}\left[\frac{1}{s}L\left[u\frac{\partial u}{\partial x}\right]\right] \tag{81}$$

Writing the solution term as a decomposition series, we have the following.

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

Plugging the above into the given problem gives the form.

$$\sum_{n=0}^{\infty} u_n(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - \left(L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} u_n(x, t)]\right]\right) \tag{82}$$

where A_n are the so-called Adomian polynomials that represents the nonlinear terms. The first few components are given as

$$\begin{aligned} A_0(u) &= u_0 u_{0x} \\ A_1(u) &= u_0 u_{1x} + u_1 u_{0x} \\ A_2(u) &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \\ A_3(u) &= u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x} \\ &\vdots \end{aligned} \tag{83}$$

Comparing the coefficients of both sides, we have the expressions for the iterative solutions given as

$$u_0(x, t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}$$
$$u_1(x, t) = -L^{-1}\left[\frac{1}{s}L[H_0(u)]\right] = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} + \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{98} \tag{84}$$

$$u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[H_1(u)]\right]$$
$$= \frac{5t^{12}}{8064} + \frac{2xt^{11}}{2079} + \frac{2783t^{10}}{302400} + \frac{38xt^9}{2835} + \frac{143t^8}{2880} + \frac{22xt^7}{315} + \frac{7t^6}{12} + \frac{2xt^5}{15}$$

Cancelling out the noise terms in the components, $u_0(x,t)$ and $u_1(x,t)$. The exact solution of the problem become.

$$u(x,t) = t^2 + xt \tag{85}$$

Example 4. Consider the following diffusion-convection problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \tag{86}$$

with the initial conditions, $u(x,0) = x + e^{-x}$

Taking the Laplace transform of both sides of the given equation and invoking the initial conditions gives the form.

$$u(x,s) = \frac{x+e^{-x}}{s} + \frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u\right] \tag{87}$$

Taking the inverse Laplace transform of both sides, we obtain.

$$u(x,t) = x + e^{-x} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u\right]\right] \tag{88}$$

Next, we apply the Adomian decomposition procedure, we get the form.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Plugging the above expression into the inverse equation, we have the form.

$$\sum_{n=0}^{\infty} u_n(x,t) = x + e^{-x} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} u_n(x,t)\right]\right] \tag{89}$$

Comparing the coefficients of like powers on both sides, we have

$$u_0(x,t) = x + e^{-x}$$

$$u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_0}{\partial x^2} - u_0(x,t)\right]\right] = -xt$$
$$u_2(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_1}{\partial x^2} - u_1(x,t)\right]\right] = x\frac{t^2}{2!} \quad (90)$$
$$u_3(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_2}{\partial x^2} - u_2(x,t)\right]\right] = -x\frac{t^3}{3!}$$
$$\vdots$$

And so on, therefore the series solution is given in the form.

$$u(x,t) = e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

which converges rapidly to the exact solution given as

$$u(x,t) = e^{-x} + xe^{-t} \quad (91)$$

Example 5. Consider the inhomogeneous nonlinear Klein-Gordon equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x\cos t + x^2\cos^2 t \quad (92)$$

with the initial condition, $u(x,0) = x, u_t(x,0) = 0$

Taking the Laplace transform of both sides of the given equation subject to the initial condition and rearranging gives.

$$u(x,s) = \frac{x}{s} - \frac{x}{s(1+s^2)} + \frac{(2+s^2)x^2}{s^3(4+s^2)} + \frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u^2\right] \quad (93)$$

Taking the inverse Laplace transform gives the expression.

$$u(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \frac{1}{4}x^2t^2 + \frac{x^2}{8} + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]\right] \quad (94)$$

Next, we apply the standard Adomian decomposition procedure, we have.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Plugging into the given equation gives the expression in terms of the unknown term as.

$$\sum_{n=0}^{\infty} u_n(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \frac{1}{4}x^2t^2 + \frac{x^2}{8} + \left(L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} A_n(u)\right]\right]\right) \quad (95)$$

where $A_n(u)$ represent the so-called Adomian polynomials. The first few of them are defined as

$$A_0(u) = u_0^2$$
$$A_1(u) = 2u_0u_1$$
$$A_2(u) = 2u_0u_2 + u_1^2$$
$$A_1(u) = 2u_0u_3 + 2u_1u_2$$

Comparing the coefficients of both sides of the solution term, we have the iterative approximations given as

$$u_0(x,t) = x\cos t - \frac{1}{8}x^2\cos t + \left(\frac{xt}{2}\right)^2 + \frac{x^2}{8}$$
$$u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u_0}{\partial x^2} - H_0(u)\right]\right] = \frac{1}{8}x^2\cos t - \left(\frac{xt}{2}\right)^2 - \frac{x^2}{8} + \frac{1}{64}\cos 2t + \cdots \quad (96)$$

Cancelling out the noise terms in the $u_0(x,t)$ and $u_1(x,t)$, then the remaining terms of $u_0(x,t)$ which satisfies the given problem gives the solution as

$$u(x,t) = x\cos t \quad (97)$$

Example 6. Consider the following inhomogeneous nonlinear PDE given as

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 2x + t^4 \quad (98)$$

with the initial conditions, $u(x,0) = 0, u_t(x,0) = 2$

Taking the Laplace transform of both sides of the given problem subject to the initial condition and rearranging gives the resulting expression.

$$u(x,s) = \frac{2}{s^2} + \frac{2x}{s^3} + \frac{24}{s^7} - \frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \tag{99}$$

Taking the inverse Laplace transform of both sides, we have.

$$u(x,t) = 2t + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]\right] \tag{100}$$

Next, we apply the Adomian decomposition method by writing the unknown as a series of the form.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Substitution into the above expression gives the expression.

$$\sum_{n=0}^{\infty} u_n(x,t) = 2t + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n(u)\right]\right] \tag{101}$$

The first few terms of the Adomian polynomials denoted $A_n(u)$ are given by the expressions.

$$A_0(u) = \left(\frac{\partial u_0}{\partial x}\right)^2 = t^4$$

$$A_1(u) = 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} = 0$$

$$A_2(u) = \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} = 0$$

$$A_3(u) = 2\frac{\partial u_0}{\partial x}\frac{\partial u_3}{\partial x} + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} = 0$$

Comparing the coefficients of like powers of the solution term, the iterative approximations become

$$u_0(x,t) = 2t + xt^2 + \frac{t^6}{30}$$

$$u_1(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u_0}{\partial x^2} + H_0(u)\right]\right] = -\frac{t^6}{30} \tag{102}$$

$$u_2(x,t) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u_1}{\partial x^2} + H_1(u)\right]\right] = 0$$

$$\vdots$$

$$u_k(x,t) = 0, k \geq 2$$

Therefore, the exact solution of the problem is given as follows.

$$u(x,t) = 2t + xt^2 \tag{103}$$

6. Results and Discussion

Table 1. Comparison between Exact, LADM and HPTM solutions for Example 1 with $t = 1$.

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	1.54308	1.543071	1.543070	0.000009	0.0000010
0.10	1.64291	1.642900	1.642890	0.000001	0.0000001
0.20	1.74175	1.741745	1.741744	0.000056	0.0000001
0.30	1.838591	1.838590	1.838590	0.000009	0.0000001
0.40	1.93250	1.93249	1.932481	0.000001	0.0000001
0.50	2.02251	2.02250	2.022500	0.000001	0.0000001
0.60	2.10772	2.10771	2.107701	0.000001	0.000002
0.70	2.18730	2.18729	2.187280	0.000001	0.000002
0.80	2.260400	2.260398	2.260390	0.000049	0.000010
0.90	2.32640	2.32640	2.326391	0.000001	0.000009

1.00	2.38552	2.385510	2.385509	0.000001	0.0000001

Table 2. Comparison between Exact, LADM and HPTM solutions for Example 2 with $t = 1$

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	0.00000	0.00000	0.000000	0.00000	0.00000
0.10	2.71828	2.71828	2.71828	0.00000	0.00000
0.20	3.84423	3.84423	3.84423	0.00000	0.00000
0.30	4.70820	4.70820	4.70820	0.00000	0.00000
0.40	5.43656	5.43656	5.43656	0.00000	0.00000
0.50	6.07826	6.07826	6.07826	0.00000	0.00000
0.60	6.65840	6.65840	6.65840	0.00000	0.00000
0.70	7.19190	7.19190	7.19190	0.00000	0.00000
0.80	7.688460	7.688460	7.688460	0.00000	0.00000
0.90	8.154850	8.154850	8.154850	0.00000	0.00000
1.00	8.59596	8.59596	8.59596	0.00000	0.00000

Table 3. Comparison between Exact, LADM and HPTM solutions for Example 3 with $t = 1$

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	1.0000	1.00000	1.00000	0.00000	0.00000
0.10	2.0000	2.00000	2.00000	0.00000	0.00000
0.20	3.0000	3.00000	3.00000	0.00000	0.00000
0.30	4.0000	4.00000	4.00000	0.00000	0.00000
0.40	5.0000	5.00000	5.00000	0.00000	0.00000
0.50	6.0000	6.00000	6.00000	0.00000	0.00000
0.60	7.00000	7.00000	7.00000	0.00000	0.00000
0.70	8.0000	8.00000	8.00000	0.00000	0.00000
0.80	9.0000	9.00000	9.00000	0.00000	0.00000
0.90	10.0000	10.0000	10.0000	0.00000	0.00000
1.00	11.0000	11.0000	11.0000	0.00000	0.00000

Table 4. Comparison between Exact, LADM and HPTM solutions for Example 4 with $t = 1$

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	1.00000	1.00000	1.00000	0.00000	0.00000
0.10	0.735759	0.735759	0.735759	0.00000	0.00000
0.20	0.871094	0.871094	0.871094	0.00000	0.00000
0.30	1.153430	1.153430	1.153430	0.00000	0.00000
0.40	1.489830	1.489830	1.489830	0.00000	0.00000
0.50	1.846140	1.846140	1.846140	0.00000	0.00000
0.60	2.209760	2.209760	2.209760	0.00000	0.00000
0.70	2.576070	2.576070	2.576070	0.00000	0.00000
0.80	2.943370	2.943370	2.943370	0.00000	0.00000
0.90	3.311040	3.311040	3.311040	0.00000	0.00000
1.00	3.678844	3.678844	3.678844	0.00000	0.00000

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Table 5. Comparison between Exact, LADM and HPTM solutions for Example 5 with $t = 1$

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	0.00000	0.00000	0.00000	0.000000	0.000000
0.10	0.054032	0.054032	0.054032	0.000000	0.000000
0.20	0.108060	0.108060	0.108060	0.000000	0.000000
0.30	0.162091	0.162091	0.162091	0.000000	0.000000
0.40	0.216121	0.216121	0.216121	0.000000	0.000000
0.50	0.270151	0.270151	0.270151	0.000000	0.000000
0.60	0.324181	0.324181	0.324181	0.000000	0.000000
0.70	0.378211	0.378211	0.378211	0.000000	0.000000
0.80	0.432242	0.432242	0.432242	0.000000	0.000000
0.90	0.486272	0.486272	0.486272	0.000000	0.000000
1.00	0.540302	0.540302	0.540302	0.000000	0.000000

Table 6. Comparison between Exact, LADM and HPTM solutions for Example 6 with $t = 1$

x	Approximate Solution			Error Analysis	
	Exact Solution	HPTM	LADM	HPTM	LADM
0.00	0.00000	0.00000	0.00000	0.000000	0.000000
0.10	0.054032	0.054032	0.054032	0.000000	0.000000
0.20	0.108060	0.108060	0.108060	0.000000	0.000000
0.30	0.162091	0.162091	0.162091	0.000000	0.000000
0.40	0.216121	0.216121	0.216121	0.000000	0.000000
0.50	0.270151	0.270151	0.270151	0.000000	0.000000
0.60	0.324181	0.324181	0.324181	0.000000	0.000000
0.70	0.378211	0.378211	0.378211	0.000000	0.000000
0.80	0.432242	0.432242	0.432242	0.000000	0.000000
0.90	0.486272	0.486272	0.486272	0.000000	0.000000
1.00	0.540302	0.540302	0.540302	0.000000	0.000000

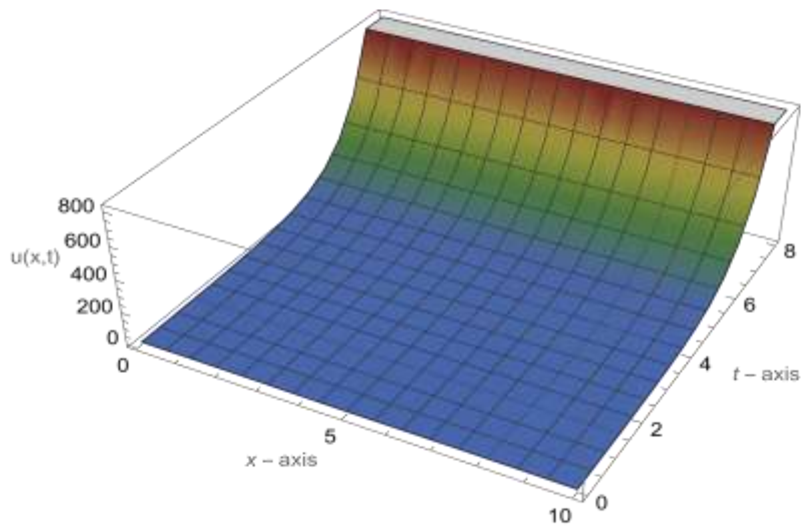

Figure 1. 3D plot of Example 1 for $0 \leq x \leq 10, 0 \leq t \leq 8$

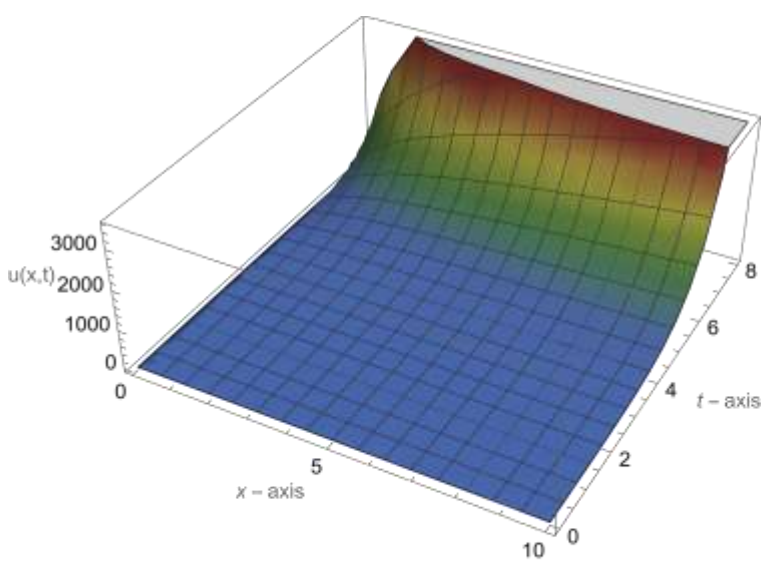

Figure 2. 3D plot of Example 1 for $0 \leq x \leq 10, 0 \leq t \leq 8$

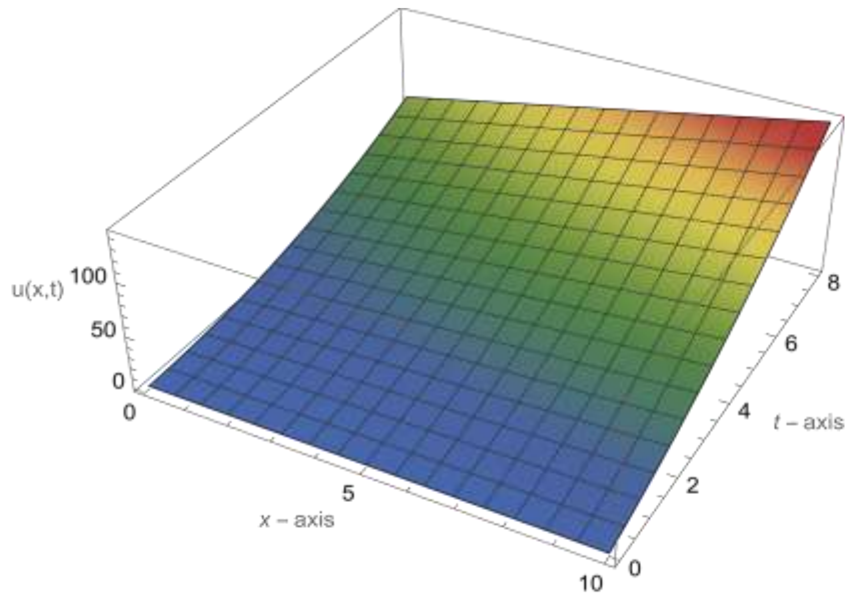

Figure 3. 3D plot of Example 3 for $0 \leq x \leq 10, 0 \leq t \leq 8$

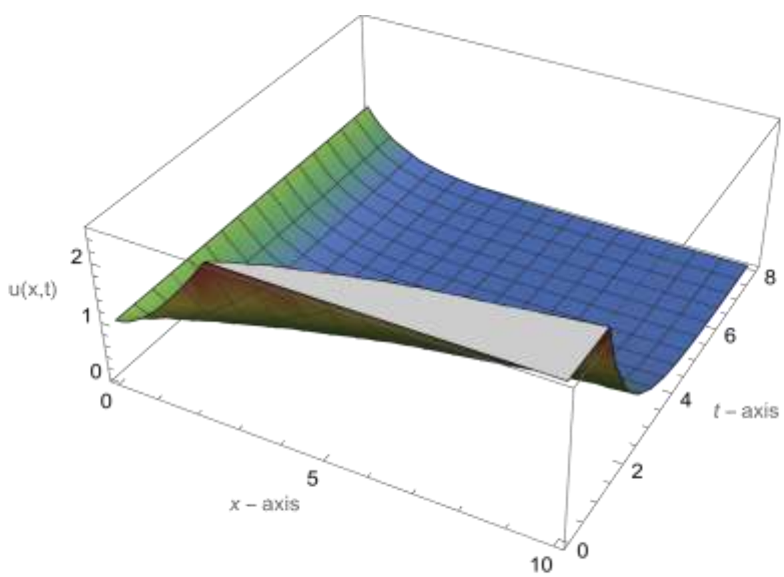

Figure 4. 3D plot of Example 4 for $0 \leq x \leq 10, 0 \leq t \leq 8$

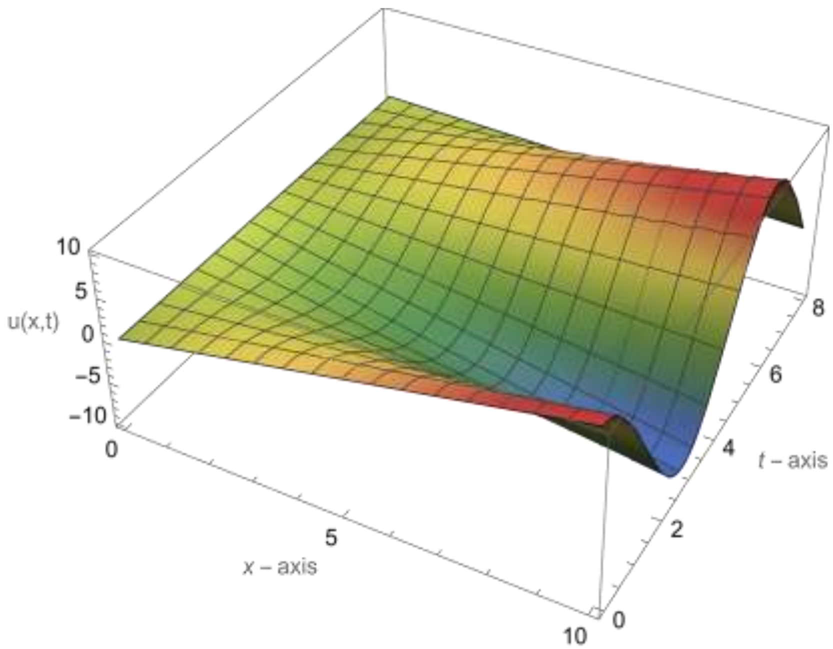

Figure 5. 3D plot of Example 5 for $0 \leq x \leq 10, 0 \leq t \leq 8$

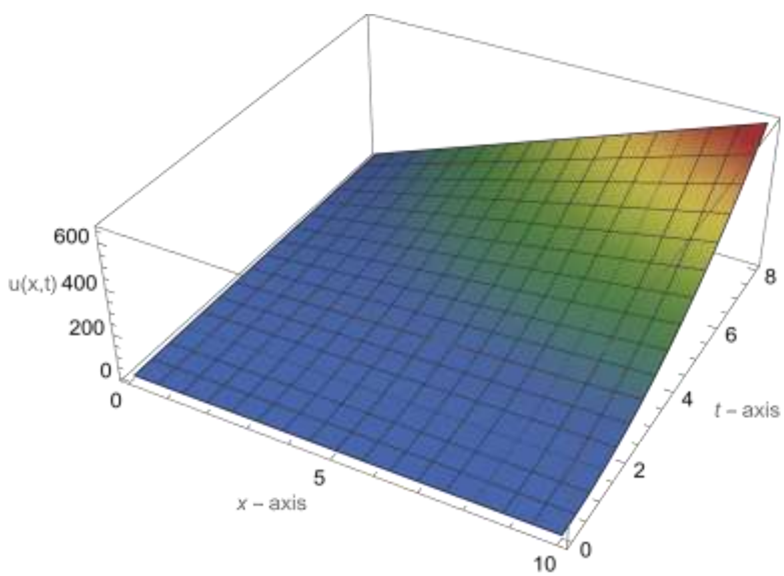

Figure 6. 3D plot of Example 6 for $0 \leq x \leq 10, 0 \leq t \leq 8$

7. Conclusion

In this research article, we presented a modification of the famous Adomian decomposition standard homotopy permutation method by coupling them with Laplace transform method for solving nonlinear partial differential equations frequently encountered in the physical sciences. In these methods, the nonlinear terms are obtained using the so-called Adomian polynomials and He's polynomials which overcome the inherent deficiency of the inability of Laplace transform method to treat nonlinear problems. The proposed technique is implemented to solve the linear and nonlinear Kleidon-Gordon equations, linear transverse equation of a vibrating beam, homogeneous and inhomogeneous nonlinear PDEs, advection equation, diffusion-convection and KDV equation respectively. From the considered problems, the presented techniques prove effective and generated numerical results which converges rapidly to the closed form solution. The findings also revealed that, all the nonlinear PDEs produced the noise terms, whereas they are absent for the linear PDEs.

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