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Topological Method for the Existence of Weak Solution of Boundary Value Problem of Ordinary Differential Equations

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Abstract: *This paper investigates the existence of weak solutions of boundary value problems associated with ordinary differential equations of order 2k by means of topological method. The approach adopted is purely analytical and does not rely on numerical computations; instead, it focuses on the qualitative behavior of the solution. The primary aim is to establish the existence of weak solutions of even-order boundary value problems and to examine their relationship with classical solutions. The principal contribution of this work is the proof of a modified version of Félix Browder's existence theorem, which provides sufficient conditions for the existence of weak solution to these problems. Several examples are presented to demonstrate the effectiveness and applicability of the main theorem.*

Keywords: topological method, existence of weak solutions, boundary value problem

INTRODUCTION

This study is based on topological methods for establishing the existence of weak solutions to boundary value problems (BVPs) of ordinary differential equations. A boundary value problem consists of a differential equation together with boundary conditions, and a solution to a boundary value problem is a function that satisfies both the equation and the prescribed conditions. One important class of boundary value problems is the Sturm–Liouville problem, whose analysis involves eigenvalues of differential operators (Pohyanin, 2002). Boundary value problems for ordinary and partial differential equations arise frequently in engineering and applied sciences. Many of these problems do not admit closed-form solutions due to the complexity of the boundary conditions and the nonlinearity involved. As a result, various approaches have been developed to study such problems, including numerical methods and topological methods. Topological methods rely on the concept of topological invariants and existence theorems to establish the existence of weak solutions (Weng & Wang, 2018). A weak solution of a boundary value problem is a function that may not possess all classical derivatives but still satisfies the differential equation in a suitably defined sense. Different notions of weak solutions exist depending on the type of equation under consideration. Weak solutions are particularly important because many differential equations arising in real-world models do not admit sufficiently smooth solutions. Even when classical solutions exist, it is often more convenient to first establish the existence of weak solutions and then prove additional regularity properties (Jost, 2013). One of the most significant approaches in the qualitative analysis of nonlinear elliptic and parabolic boundary value problems is the abstract topological method. This approach facilitates the study of solvability and bifurcation of eigenvalue and boundary value problems through the use of topological invariants and existence theorems (Weng & Wang, 2018). Continuation theorems for boundary value problems of ordinary differential equations are commonly based on reformulating the problem as an operator equation in an abstract space and applying degree theory. The most fundamental framework in this context is the Leray–Schauder degree theory for compact perturbations of the identity in Banach spaces (Abbassi, 2021). In many cases, boundary value problems for ordinary differential equations naturally lead to operator equations consisting of the sum of a Fredholm linear operator of index zero and a nonlinear operator that is relatively compact with respect to the linear part (Mawhin, 2014). The origins of this method trace back to the work of Leray and Schauder, who extended Brouwer's degree theory to infinite-dimensional Banach spaces by introducing the degree of mappings of the form $I - T: X \to X$, where I is the identity and T is a completely continuous operator (Abbassi, 2021). They demonstrated that establishing a priori bounds on solutions is a crucial step in proving existence results for nonlinear boundary value problems. The Leray–Schauder method has been widely and successfully applied to nonlinear differential, integral, and functional equations, with notable applications in fluid mechanics and elliptic partial differential equations. However, its applicability to more general boundary value problems for ordinary differential equations is limited, as such problems often require stronger restrictions, particularly for general Dirichlet conditions (Eljaneid, 2017). To address these limitations, researchers have developed alternative topological approaches for more general classes of operator equations in Banach spaces. Agarwal et al. (2003) introduced two topological methods for studying the solvability of nonlinear boundary value problems and differential inclusions on finite intervals. The first method generalises earlier results by the Florence group of mathematicians and removes restrictive assumptions such as convexity conditions. The second method concerns the existence of bounded solutions on the positive real axis and extends results previously obtained for periodic problems (Chu & Nieto, 2009). Several numerical and analytical approaches have also been proposed. Nakao (2001) developed a computer-assisted verification technique for nonlinear two-point boundary value problems based on a Newton-type operator and Sadovskii's fixed point theorem. Serdal (2009) compared several numerical methods, including the Adomian decomposition method, homotopy perturbation method, and variational iteration method, noting that these approaches yield approximate rather than exact solutions. Consequently, topological existence theorems remain essential for establishing rigorous solution results. Recent studies have extended topological methods to fractional differential equations, weak topologies, and Banach space settings. Researchers such as Benchohra and Seba (2009), Ezzinbi and Taoudi (2015), and

Chlebowicz and Taoudi (2017) have employed measures of weak non-compactness and fixed point theorems to obtain existence results. These approaches are particularly effective in reflexive Banach spaces, where bounded sets are relatively weakly compact. The theory of time scales, introduced by Hilger (1990), unifies continuous and discrete analysis and has generated growing interest in the study of boundary value problems for dynamic equations. Existence results for impulsive and second-order boundary value problems on time scales have been obtained using fixed point and Leray–Schauder type methods (Benchohra et al., 2004). However, the application of critical point theory in this context remains limited. Finally, Afrouzi et al. (2016) established the existence of at least one weak solution for a three-point boundary value problem of Kirchhoff type, further demonstrating the effectiveness of topological and variational methods in the study of nonlinear boundary value problems.

MATHEMATICAL ANALYSIS

In the classical treatment of differential equations, the solution and its derivatives are required to be continuous functions. One therefore works in space $C^k(\bar{\Omega})$ that contains functions with continuous derivatives up to order k or in spaces where these derivatives are Holders continuous. When the strong form of a differential equation is replaced by a variational formulation, then instead of point wise differentiability, one needs to only ensure the existence of some integrals that contain the unknown function as certain derivatives. Thus it makes sense to use function spaces that are especially suited to this situation. However, some basic fundamental results from functional analysis are presented to facilitate the understanding of the main theorem of this article. Let Ω be a bounded domain in n-dimensional Euclidean space R^n, n > 1, points in R^n are given by $x = (x_1, x_2, \dots, x_n)$, when n = 2 the points are given as (x, y). Let $\partial\Omega$ be the boundary of the domain Ω, then assuming that for all points $x \in \partial\Omega$, the unit vector of the outward normal v to $\partial\Omega$ is defined as v = $(v_1, v_2, \dots, v_n)$. When n = 2, the components of the vector in the outward normal will be denoted by $v_{x,} v_y$ as v = $(v_{x,} v_y)$, beside the derivative in the direction of the outward normal of the function u i.e.

$$\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} v_x + \frac{\partial u}{\partial y} v_y \quad (1)$$

Therefore one introduce the derivative in the direction of the tangent as;

$$\frac{\partial u}{\partial \tau} = -\frac{\partial u}{\partial x} v_x + \frac{\partial u}{\partial y} v_y \quad (2)$$

Suppose that $u = u(x_1, x_2, \dots, x_n)$ be a function defined on the domain $\Omega \subset R^n$, the gradiant of R^n, is an n-dimensional vector function denoted by *gradu* and is defined as $gradu = \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}$. When n=2, then the function u (x, y) is defined as $gradu = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ and $|gradu|^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2$. Let Ω be a domain in $R^n, n \geq 1$ and let $a_i = a_i(x; \eta_0, \eta_{1,} \dots, \eta_n) \in R^{n+1}$ (Zacher, 2009).

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Definition 1. Suppose that *f = f(x)* be a function defined on the domain Ω. Consider the differential equation of the form.

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_i(x,u,gradu)+\ a_0(x,u,gradu)=f(x) \tag{3}$$

Equation (3) is called differential equation of the second order in divergent form, the function a_i is called the coefficient of (3) where the function *f(x)* is the right hand side of the of equation (3).

Suppose n = 2 then (3) becomes;

$$-\frac{\partial}{\partial x}a_i\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)+\ \frac{\partial}{\partial x}a_2\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}a_0\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=f(x) \tag{4}$$

When the coefficient a_i (i = 0, 1, 2) is defined for (x, y) ∈ Ω and $\left(\eta_0,\eta_{1,}\eta_{2,}\right)\ \in\ R^3$, the most general form of the equation of second order is defined as

$$F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial y\partial x},\frac{\partial^2 u}{\partial y^2}\right)=0 \tag{5}$$

Where F = F($x,y,\eta_0,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6$) is a given function of nine variables defined for *(x,y)*∈ $\Omega$and ($\eta_0,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6$) ∈ R^3. Equation (5) is a special case of equation (4) which is a general second order differential equation in two variables.

Definition 2.

$$\text{Let } \sum_{|\alpha|\le k}(-1)^{|\alpha|}D^\alpha a_\alpha\big(x;\delta_k u(x)\big)=f(x) \tag{6}$$

Be the formal differential equation of order 2k in divergent form, where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\ for\ \alpha_1\ \ge 0;and\ i=(1\ldots n)$ is called the n-dimensional multi-index. Where $|\alpha|=\sum_{i=1}^{n}\alpha_i$ is the length of the multi-index. The symbol $D^\alpha u$ is given by $D^\alpha u=\frac{\partial^{|\alpha|}u}{\partial x_1^\alpha,\partial x_2^\alpha\ldots\ldots\partial x_n^\alpha}$ the partial derivative of a function u = u(x) defined on the domain $\Omega\ \subset\ R^n$. Also the symbol $\delta_k u(x)$ in equation (6) is a vector function whose components are all the derivative of the function u of order 0, 1, 2, …, k and is given by $\delta_k u(x)=\left(u,\frac{\partial u}{\partial x},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1^2},\ldots,\frac{\partial^k u}{\partial x_1^k},\ldots,\frac{\partial^k u}{\partial x_n^k}\right)$, suppose k =1 then

$$\delta_1 u(x)=\left(u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\ldots,\frac{\partial u}{\partial x_n}\right)\text{ etc.}$$

Definition 3. The classical solution of boundary value problem $\phi_i\big(x;\ \delta_k u(x)\big)=0$ is the function u = u(x) defined for $\overline{\Omega}=\ \Omega\ \cup\ \partial\Omega$ and satisfies the following conditions.(i)u ∈ $C^{2k}(\overline{\Omega})$. (ii) (5) is satisfied for all x ∈ Ω. (iii) The boundary conditions $\Phi_i\big(x;\ \delta_k u(x)\big)=0$ are satisfied for all x ∈ ∂Ω.

If $a_i \in C^{|\alpha|}(\bar{\Omega} \times R^m)$, then the function T = T(x) defined for $x \in \Omega$ by the formula;

$$T(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^{\alpha} a_{\alpha}(x; \delta_k u(x)) \tag{7}$$

is continuous that is T $\in C^0(\bar{\Omega}), if\ u \in C^{2k}(\bar{\Omega})\ then$ (1.5) defines differential operator A which assign a function $T = Au$ from $C^0(\bar{\Omega})$.

Definition 4.: Let k be a positive integer and let $p \geq 1$. The Sobolev space $W^{kp}(\Omega)$ is the set of all functions $u \in L_p(\Omega)$ all od whose generalized derivatives $D^{\beta}u$ of order at most k most exist and again lie in $L_p(\Omega)$, The expression $\|u\|_{k,p} = \sum_{|\beta| \leq k} \|D^{\beta}u\|_p \equiv \|u\|_{kp0} = \left(\sum_{|\beta|=k} \|D^{\beta}u\|_p^p\right)^{1/p}$ Defined a norm on the space $W^{kp}(\Omega)$, and $W_0^{kp}(\Omega)$. With these norms. $W^{kp}(\Omega)$ and $W_0^{kp}(\Omega)$ are separable Banach space.

Definition 5 . Let U be a linear space. A mapping $\|\cdot\| : U \rightarrow R$ is said to be a norm if the following conditions are satisfied. (N_i): $\|u\| \geq 0\ \forall\ u \in U, \|u\| = 0\ if\ and\ only\ if\ u = 0. (N_{ii})$: $\|\lambda u\| = |\lambda| \|u\|\ \forall\ \lambda \in R, u \in U. (N_{iii})$: $\|u + v\| \leq \|u\| + \|v\|, \forall\ u, v \in U, the\ pair$ $(\|\cdot\|, R)$ is called a norm linear space.

Definition 6. A sequence u^k in a normed linear space is said to be Cauchy sequence if for each e > 0 there exist N€ such that $\|u^k - u^l\| \leq e, \forall\ k, l \geq N(e)$.

Definition 7. A normed linear space is said to be complete if every Cauchy sequence $u^k \in U\ converges\ in\ U$. i.e. there exist a $u_0 \in U\ with\ \lim_{k\to\infty} \|u^k - u_0\| = 0\ equivalently\ u_0 - \lim_{k\to\infty} u^k$.

Definition 8. A complete normed linear space is called a Banach space.

The property state above is of fundamental importance for both in existence theorem for solutions of variational problems and in proof of convergence of numerical methods (Grossmann, *et al,* 2007).

Definition 9. Let U, V be two normed linear space with norms $\|\cdot\| U\ and\ \|\cdot\|\ V$ respectively, a mapping T: U $\rightarrow V$ is continuous at u $\in$ U if for any sequence $u^k \in$ U converging to u then $\lim_{k\to\infty} Tu^k = Tu$.

i.e. $\lim_{k\to\infty} \|u^k - u_0\|_u = 0\ implies\ that\ \lim_{k\to\infty} \|Tu^k - Tu_0\|_v = 0$.

A mapping is continuous if it is continuous at every point.

Let Ω be a domain in R^n which is of class $C^{0,1}$ and let g and u be two functions from $C^1(\bar{\Omega})$, such that the relations.

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$$\int_{\Omega} \frac{\partial g(x)}{\partial x_i} v dx = - \int_{\Omega} g(x) \frac{\partial v(x)}{\partial x_i} dx + \int_{\partial\Omega} g(x)v(x)v_i(s)ds \quad (8)$$

Remark If n= 1, $\Omega = (a, b) and\ \partial\Omega\ consist\ of\ x = a\ and\ x = b$ then (8) becomes

$\int_a^b g'(x)v\, dx = - \int_a^b g(x)\, v'(x) + [g(b)v(b) - g(a)v(a)]$. If m is any

positive integer and $g, v\ \in C^m(\bar{\Omega})\ then$ (8) $becomes$:

$$\int_{\Omega} D^{\alpha} g(w)v(x)dx = (-)^{|\alpha|} \int_{\Omega} g(x)D^{\alpha}v(x)\, d + \int_{\partial\Omega} G(g, w)ds \quad (9)$$

holds $= m, G(g, w) \equiv$ $g(x)v_i(x)$ 2, and $|\beta| < m, |\gamma| <$ m, $v_i = v_i(x)$ is the component of the outward normal. Observe that if $\Omega\ \in C^{0,1}, v \in C_0^m(\Omega)$ and $g \in C^m(\bar{\Omega})$.

$$\int_{\Omega} D^{\alpha} g(w)v(x)dx = (-1)^{|\alpha|} \int_{\Omega} g(x)D^{\alpha}v(x)\, dx \quad (10)$$

Remark

If $u\ \in C^{2k}(\Omega)$ is a classical solution of

$$\sum_{|\alpha|}(-1)^{|\alpha|} D^{\alpha} a_{\alpha}\big(x; \delta_k u(x)\big) = f(x) \quad (11)$$

And let $v \in\ C_0^k(\Omega)$, then multiplying (3.4) by v(x) and integrating the resulting Equation one obtain.

$$\sum_{|\alpha| \le k}(-1)^{|\alpha|} \int_{\Omega} D^{\alpha} a_{\alpha}\big(x; \delta_k u(x)\big)v(x)\, dx = \int_{\Omega} f(x)v(x)dx \quad (12)$$

Now let g(x) = $a_{\alpha}\big(x; \delta_k u(x)\big)$ and using (3.3) we have

$$\int_{\Omega} D^{\alpha} a_{\alpha}\big(x; \delta_k u(x)\big)v(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} a_{\alpha}\big(x; \delta_k u(x)\big)D^{\alpha}v(x)dx$$

Combining these Equations, one obtain:

$$\sum_{|\alpha| \le k} \int_{\Omega} a_{\alpha}\big(x; \delta_k u(x)\big)D^{\alpha}v(x)\, dx = \int_{\Omega} f(x)v(x)dx \quad (12\ )$$

which is valid for every function $v\ \in\ C_0^k(\Omega)$. This show that if u is the classical solution of (10) then the integral identity (8) holds for every function $v\ \in\ C_0^k(\Omega)$.

Definition 10 : Let $L_p\ (\Omega)\ be\ a\ domain\ in\ R^n\ \ and\ let\ h = h(x, \eta)$ be the function defined for almost all $x\ \ \in\ \Omega$ and for all $\eta \in R^m$, then the function h has a caratheodory property if: (i): $for\ all\ \eta \in R^m$, the function $h_{\eta}(x) = h(x, \eta)$ as a function of variable x is measurable on Ω. (ii): for almost all $x\ \ \in\ \Omega$, the function $h_x(\eta) = h(x, \eta)$ as a function of variable η is continuous on R^m and is denoted as $h \in CAR$. This means that the function h satisfies the following inequality where it is assumed that the function $h = a_{\propto}(x, \eta)$.

$$|a_{\propto}(x,\eta)| \leq g_{\propto}(x) + c_{\propto} \sum_{|\propto| \leq k} |\eta|^{p-1} \qquad (13)$$

Where $g_{\propto}(x)$ is a given function from $L_q(\Omega) and\ c_{\propto}\ is\ non-negative\ constant, \frac{1}{p}+\frac{1}{q}=1.$

The inequality (13) tell us how the function $a_{\propto}(x,\eta)$ must behave with respect to the variable η by setting certain constrain on its growth i.e. the function $a_{\propto}$ cannot grow fast in the variable η than the polynomial $c_1 + c_2|\eta|^{p-1}$. one can call the function $a_{\propto}$, which satisfy the inequality of 3.9 function with polynomial growth and is denoted by $a_{\propto} \in CAR(p)$.

Recall the formal differential operator of order *2k* defined by;

$$(Tu)(x) = \sum_{|\propto| \leq k} (-1)^{|\propto|} D^{\propto} a_{\propto}\big(x;\ \delta_k u(x)\big) \qquad (14)$$

With coefficients $a_{\propto} \in CAR(p)$, where $|\propto| \leq k, p > 1$. Let *f* be a continuous linear functional on $\Big((W)_0^{k,p}(\Omega)\Big)^*$ and let φ be the function from $W^{k,p}(\Omega)$, then the function $u \in W^{k,p}(\Omega)$ is the weak solution of the Dirichlet problem for the operator T if ; (i) $u-\varphi \in W_0^{k,p}(\Omega)$. (ii): for every $v \in W_0^{k,p}(\Omega)$ we have;

$$\sum_{|\propto| \leq k} \int_{\Omega} a_{\propto}\big(x;\ \delta_k u(x)\big) D^{\propto} v(x))dx = \langle f, v \rangle \qquad (15)$$

Condition (i) states that $D^{\beta}(u-\varphi)|_{\partial}\Omega = 0$ or in order words $D^{\beta}u = D^{\beta}\varphi$ on the boundary, in the sense of traces, since the function φ is given so also the traces $D^{\beta}\varphi|_{\partial}\Omega$. That is, the solution u and it derivativ... $D^{\beta}u$ of order up to $k-1$ assume the prescribed values on $\partial\Omega$. Condition (ii) states that the function *u* is the solution of the formal differential equation *Tu(x)* = $f\ on\ \Omega$.

For instance, consider the case *N=1*, $\Omega = (0,1) and\ k = 1$ then the formal differential operator *T* is the following ODE operator of second order.

$$(Tu)(x) = -\frac{d}{dx} a_1\big(x, u(x), u'(x)\big) + a_0\big(x, u(x), u'(x)\big)$$

Where it is assumed that $a_i \in CAR(p)\ for\ (i = 0,1) that\ is\ |a_i(x,\eta_0.\eta_1)| < g_i(x) + c_i(|\eta_0|^{p-1} + |\eta_1|^{p-1})$ where (i = 0, 1) with constant $c_i \geq 0$ and the function $g \in L_q(\Omega)$. The function $u \in W^{1,P}(0,1)$ is the weak solution of the Dirichlet problem for the operator *T* if (i): $u - \varphi \in W_0^{1,P}(0,1)$, (ii): for every function $v \in W_0^{1,P}(0,1)$, one obtain

$$\int_0^1 \big|a_i\big(x, u(x), u'(x)\big)v'(x) + a_0\big(x, u(x), u'(x)\big)\, v(x)\big|\, dx = \langle f, v \rangle$$

Where φ is a given function from $W^{1,P}(0,1) and\ f\ is\ a\ given\ function\ from\ W_0^{1,P}(0,1)^*$. Now choose the functional as follow

$$\langle f, v \rangle = \int_0^1 f(x)v(x)dx\ \forall\ v \in W_0^{1,P}(0,1)$$

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And given the function φ in the form $\varphi(x) = c_0 + (c_1 - c_0)x \; for \; u - \varphi \in W_0^{1,P} \Rightarrow u(0) - \varphi(0) = 0, u(1) - \varphi(1) = 0 \Rightarrow u(0) = c_0, u(1) = c_0$ which means the function u satisfies the given boundary values condition.

Theorem: Banach contraction states that If T^n is a contraction, then T: X → X for all n ≥ 1, the T has a unique fixed point, and this is known as generalized Banach space.

3.0 Existence and Uniqueness Theorem

Let X be a complete normed Linear space, then a mapping T: X→ X is called contraction if $\forall\, x, y \in X.\ \|Tx - Ty\| \le L\|x - y\| \; \forall\, x, y \in X \; if \; L < 1$, Then is a contraction Banach fixed point theorem also known as Banach Contraction principle state that, if T:X→ $X\ is\ a\ contraction, then\ T\ has\ a\ unique\ fixed\ point\ X = Tx$. Furthermore, the sequence $\{x_n\}_{n=1}^{\infty}$ is defined by $x_n = Tx_{n-1}, x_o - arbitrary\ and\ n > 1, 2, \ldots$ convergence to the unique fixed point *x* of *T*. Let $\frac{dy}{dt} = f(x, y), y(x_o) = x_o$ (IVP).

Theorem: Let f(x,y) be continuous functions defined on a domain $D \subseteq R^2$ such that f is Lipschitz continuous with respect to the second variable on D, then there exist a unique fixed point of the initial value problem on the internal

$|x - x_o| \le L\ where\ L\ is\ min\left(a, \frac{b}{m}\right) and\ M\ is\ max\ f(x, t). R = \{(x, t)\big||x - x_o| \le a, |y - y_o| \le b\big|\} \; \forall\,(x, y) \in R\ and\ let\ L\ be\ the\ Lipschitz\ constant\ of\ f(x, t)$, furthermore, the unique solution can be computed from the Piccard successive iteration scheme.

4.0 Browder Existence Theorem

Let U be a reflexive Banach space. Let T be an operator defined on U with values in U^* and let the following conditions be satisfied. (i) T is bounded operator, i.e. the image of any bounded subset of the space U is a bounded subset of the space U^*. (ii) the operator T is demi-continuous, i.e. for arbitrary $u_0 \in U$ and any sequence $\{u_n\}_n^{\infty}$ of element of the space U such that $u_n \to u_0\ in\ U\ then\ Tu_n \rightharpoonup Tu_0\ in\ U^*$; (iii) The operator T is coercive, i.e. $\lim_{u \to \infty} \frac{\langle Tu, u\rangle}{\|u\|} = \infty$; (iv) The operator T is monotone on the space of *U*, i.e for all $u, v \in U\ we\ have\ \langle Tu - Tv, u - v\rangle \ge 0$. Then the equation *Tu=f* has at least one solution $u \in U$ for every $u \in U^*$. If, moreover, $\langle Tu - Tv, u - v\rangle \ge 0\ is\ strict\ for\ all\ u, v \in U, if\ u \ne v$ then equation *Tu=f* has precisely one solution $u \in U$ for every $f \in U^*$.

This theorem can be modified in terms of the coefficient of the formal differential equation of order *2k* as follows:

Theorem: Let$k \in N, p > 1\ and\ suppose\ that\ the\ following\ conditions\ are\ satisfied$

(i) $a_{\propto}(x, \eta) \in CAR^*(p)$.

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(ii) (ii) $\sum_{|\propto|\leq k}[a_{\propto}(x,\eta)-a_{\propto}(x,\xi)]\,(\eta_{\alpha}-\xi_{\alpha})>0$

(iii) $\lim_{\|u\|_{p},k\to\infty}\frac{1}{\|u\|_{k},p}\sum_{|\propto|\leq k}\int_{\Omega}a_{\propto}\big(x,\delta_{k}u(x)\big)+\delta_{k}\varphi(x))\,D^{\alpha}u(x)dx=\infty$

Then the boundary value problem has at least one weak solution, furthermore, if equality holds i.e. $(\eta_{\alpha}=\xi_{\alpha})$ in condition (ii) only then, the boundary value problem has precisely one weak solution.

Proof of the Main Theorem

To prove boundedness condition of the operator T (i): one recall the function h, $h(x;\,\eta_1,\cdots,\eta_m)$ in (3.9) satisfy the following inequality.

$$|h(x;\eta_1,\cdots,\eta_m)|\leq g_{\propto}(x)+c_{\propto}\sum_{|\beta|\leq k}|\eta_{\beta}|^{\frac{p}{r}}$$

Hence choosing the function $h=a_{\propto}(x,\eta)$ relative to (3.11) and let *m=x=x(N, k); p > 1,* $p_i\geq 1$ and $r\in R, Since\,\frac{1}{p}+\frac{1}{q}=1, then\; q=\frac{p}{p-1}=\frac{p}{q}=(p-1)\;hence\; p_i/r=p/q$

therefore,

$|h(x;\eta_1,\cdots,\eta_m)|\leq g_{\propto}(x)+c_{\propto}\sum_{|\beta|\leq k}|\eta_{\beta}|^{p/r}$ becomes

$$|a_{\propto}(x;\eta_1,\cdots,\eta_m)|\leq g_{\propto}(x)+c_{\propto}\sum_{|\beta|\leq k}|\eta_{\beta}|^{p-1}$$

Where $g\in L_q(\Omega)$ and $c_{\propto}=c_{\propto}(t)=1\; hence\; a_{\propto}(x;\eta)\in CAR^{*}(p)$. This completes the prove of the first condition. Next to prove monotony condition of the operator T (ii) one recall (3.9) can take the form of

$$\langle Au,v\rangle=\sum_{|\propto|\leq k}\int_{\Omega}a_{\propto}\big(x,\delta_k u(x)\big)D^{\propto}v(x)dx$$

Then $\langle Tu-Tv,u-v\rangle=\langle A(u+\varphi)-A(v+\varphi),u-v\rangle$

$$\sum_{|\propto|\leq k}\Big|a_{\propto}\big(x,\delta_k u(x)\big)+\delta_k\varphi(x)-\Big(a_{\propto}\big(x,\delta_k v(x)\big)+\delta_k\varphi(x)\Big)\Big|\,[D^{\propto}u(x)dx-D^{\propto}v(x)dx]>0$$

$$\sum_{|\propto|\leq k}[\,a_{\propto}(x,\eta)-a_{\propto}(x,\xi)]\,(\eta_{\alpha}-\xi_{\alpha})>0$$

For all $u,v\in W^{k,p}(\Omega)$, hence the prove is complete. It remains to prove coerciveness of the operator *T.* Here, one need to show that

$$\lim_{\|u\|k,p\to\infty}\frac{1}{\|u\|_{k,p}}\sum_{|\propto|<k}\int_{\Omega}a_{\propto}\big(x,\delta_k u(x)\big)+\delta_k\varphi(x))\,D^{\alpha}u(x)dx=\infty$$

is equal to

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$\sum_{|\propto|<k}^{N} a_{\propto}(x;\ \eta)\eta_{\alpha} \geq c_1 \sum|\eta_i|^p + c_2|\eta_0|^p - c_3.\quad For\ c_1 > 0, c_2 > 0, c_3 > 0$

Let c_i $be\ a\ nonnegative\ constant\ i.e.c_i > 0\ such\ that$

$$\lim_{\|u\|k,p\to\infty} \frac{1}{\|u\|_{k,p}} \sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(\cdot)\right) + \delta_k\varphi(x))\, D^{\alpha}u(x)dx \geq c_i$$

Therefore,

$$\sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(x)\right) + \delta_k\varphi(x))\, D^{\alpha}u(x)dx \geq c_i\|u\|_{k,p}$$

$$\sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(x)\right) D^{\alpha}u(x)dx \geq c_i\|u\|_{k,p} - \sum_{|\propto|<k} \int_{\Omega} \delta_k\varphi(x))\, D^{\alpha}u(x)dx$$

$$\sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(x)\right) D^{\alpha}u(x)dx \geq c_i\|u\|_{k,p} \sum_{|\propto|<k} \left|\int_{\Omega} \delta_k\varphi(x))\, D^{\alpha}u(x)dx\right|$$

$$\sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(x)\right) D^{\alpha}u(x)dx \geq c_i\|u\|_{k,p} \sum_{|\propto|<k} \int_{\Omega} |\delta_k\varphi(x))\, D^{\alpha}u(x)|dx$$

$$\sum_{|\propto|<k} \int_{\Omega} a_{\propto}\left(x, \delta_k u(x)\right) D^{\alpha}u(x)dx \geq c_i\|u\|_{k,p} \sum_{|\propto|<k} \int_{\Omega} \|\delta_k\varphi(x))\|\|D^{\alpha}u(x)\|dx$$

$\|u\|_{k,p} = \left(\sum\|D^{\propto}u\|_p^p\right), therefore$

$$\sum_{|\propto|<k} a_{\propto}\left(x, \delta_k u(x)\right) D^{\alpha}u(x)dx \geq c_i \left(\sum\|D^{\propto}u\|_p^p\right) - \sum_{|\propto|<k}\|\eta\|\|\eta_{\alpha}\|$$

$$\sum_{|\propto|<k} a_{\propto}(x,\eta)\eta_{\alpha} \geq c_i \left(\sum|\eta_{\alpha}|^p\right) - \sum_{|\propto|<k}\|\eta\|\|\eta_{\alpha}\|$$

$hence\ let\ c_3 = \sum_{|\propto|<k}\|\eta\|\|\eta_{\alpha}\| \geq 0, (i = 1, 2)$

$$\sum_{|\propto|<k}^{N} a_{\propto}(x,\eta)\eta_{\alpha} \geq c_i \sum|\eta_{\alpha}|^p + c_2|\eta_0|^p - c_3.\ For\ c_1 > 0, c_2 > 0$$

This complete the prove!

RESULTS AND DISCUSSION

In this section, the validity of the existence theorems proved above can be used. However, different types of examples of boundary value problems are presented to illustrate the usefulness of the main theorem of this article.

Example 1

Consider the boundary value problem,

$$\begin{cases} u^{iv} + \frac{1}{\pi} arctan\, u = f\ on\ (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{cases}$$

Here, one need to show that the boundary value problem stated above has at least one weak solution. Therefore, recall the general form of ordinary differential equation of order $2k$.

$$\sum_{i=0}^{k}(-1)^i \frac{d^i}{dx^i} a_i\left(x,u,u^i,\cdots,u^k\right) = f \quad (1)$$

Since the order of the given boundary value problem is 4, hence $2k = 4$ implies that $k = 2$, this means that (i=0, 1, 2). Subtracting the value of I into (4.1) of order $2k$, one obtain

$$D^2 a_2(x;\ \eta) + a_0(x;\ \eta) = f \quad (2)$$

The given boundary value problem can be written as $u^{(iv)} + \frac{1}{\pi}\ arc\ tan\, u = \frac{d^2}{dx^2}\left(\frac{d^2u}{dx^2}\right) + \frac{1}{\pi} arctanu = f, i.e. D^2\eta_2 + \frac{1}{\pi} arc\ tan\eta_0 = f$. Now, comparing the coefficient of this equation with (4.1), this implies that $a_2(x;\ \eta) = \ \eta_2, a_1(x\!:\eta) = \frac{1}{\pi} arc\ tan\eta_0$. However, one need to show that the differential operator is bounded, this means that the coefficient $a_i(x,\eta) \in CAR(p)$ for all (i=0, 1, 2) therefore, one recall the growth condition i.e.

$$|a_i(x,\eta)|\ \le g_i(x) + c_i \textstyle\sum_{i=0} |\ \eta|^{p-1}$$

$$|a_2(x;\ \eta)| = \eta_2 \le |\eta_2|\ \le |\eta_2|^{p-1} \le\ \textstyle\sum_{i=0} |\ \eta|^{p-1} < g_2(x) + c_2 \sum_{i=0} |\eta_i|^{p-1}$$

Where $g_2(x) \equiv 0, c_2 = 1, then\ a_2(x;\ \eta) \in CAR(p)$. In the same manner.

$$|a_1(x;\eta)| = 0\ \le g_1(x) + c_1 \textstyle\sum_{i=0} |\eta_i|^{p-1}$$

Where $g_1(x) = 0, c_1 = 1, then\ a_1(x;\eta) \in CAR(p)$, for

$$|a_0(x;\eta)| = \frac{1}{\pi} arc\ tan\eta_0 \le 1 \le g_0 + c_0 \textstyle\sum_{i=0} |\eta_i|^{p-1}$$

Therefore, $a_1(x;\eta) \in CAR(p)\ for\ i = 0,1,2$. Secondly, one need to show that the operator T is monotone i.e.

$$\textstyle\sum_i^n |a_i(x;n) - a_i(x;\ \xi)|\ (\eta_i - \xi_i) \ge 0$$

$[a_2(x;\eta) - a_2(x;\ \xi)](\eta_2 - \xi_2) + [a_1(x;\eta) - a_1(x;\ \xi)](\eta_1 - \xi_1) + [a_0(x;\eta) - a_0(x;\xi)](\eta_0 - \xi_0)(\eta_2 - \xi_2)(\eta_2 - \xi_2) + 0 \times (\eta_1 - \xi_1) + (\frac{1}{\pi} arc\ tan\ \eta_0 - \frac{1}{\pi} arctan\ \xi_0)(\eta_0 - \xi_0),\ \ now\ if\ \eta_0 > \xi_0\ or\ \eta_0 < \xi_0.\ Therefore,$

$(\eta_2 - \xi_2)^2 + (\frac{1}{\pi} arctan\ \eta_0 - \frac{1}{\pi} arctan\xi_0)\ (\eta_0 - \xi_0) \ge 0$, then, the monotone condition is satisfied. Lastly, next is to show that the operator T is coersive, i.e. for $c_1 > 0, c_2 > 0, c_3 > 0\ such\ that$

$$\textstyle\sum_{i=0}^n a_i(x;\ \eta)\eta_i \ge c_1(\sum_{i=0}^n |\eta_i|^p) + c_2|\eta_0|^p - c_3$$

$$a_2(x;\eta)\eta_2 + a_1(x;\eta)\eta_1 + a_0(x;\eta)\eta_0 \ge c_1(\textstyle\sum_{i+0}^n |\eta_i|^p) + c_2|\eta_0|^p - c_3$$

$$\eta_2^2 + 0 + \frac{1}{\pi} arctan\ \eta_0^2 \ge c_1(\textstyle\sum_{i=0}^n |\eta_i|^p) + c_2\ |\eta_0|^p - c_3$$

Now choose $c_1 = \eta \sum_{i=0}^{n} |\eta_i|^{-p}, c_2 = |\eta_0|^{-p}, c_3 = 2$

Therefore, $\eta^2 + 0 + \frac{1}{\pi} arctan\eta_0^2 > 0$

$\sum_{i=0}^{n} a_i(x;\eta)\eta_i \geq 0$

This shows that the operator T is coercive. If the condition of monotonicity us strict, i.e. $\eta_0 > \xi_0 or\ \eta_0 < \xi_0$ then, the boundary value problem have precisely one weak solution. However, since all the conditions of the Theorem are satisfied then the boundary value problem under investigation have at least one weak solution.

Example 2

Let consider the Dirichlet problem defined as;

$$\begin{cases} u'' + u' + u^3 = f\ on\ (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

To investigate the weak solution of the Dirichlet problem, one recall the formal differential equation of order *2k* , i.e.

$\sum_{t=0}^{k} (-1)^i \frac{d^i}{dx^4} a_i(x, u, u', \cdots, u^k) = f\ on\ (0, 1)$

In this case, the given problem is of order 2 which implies that *k*=1, this means that *i = 0, 1,* therefore, substituting the values of *i* into formal differential equation one obtain - $Da_1(x;\eta) - (\eta_1 + \eta_0), a_0(x;\eta) = \eta_0^3$ hence one proceed to show that the coefficient of the differential operator satisfies the following conditions. For boundedness of operator, therefore one recalls that growth condition as;

$|a_i(x,\eta)| \leq g_i(x) + c_i \sum_{i=0} |\eta|^{p-1}$

$|a_i(x,\eta)| = -(\eta_1 + \eta_0) \leq |(\eta_1 + \eta_0)| \leq \sum_{i=0}^{I} |(\eta_i)| \leq \sum_{i=0}^{I} |(\eta_i)|^{p-1}$

$\leq g_i(x) + c_i \sum_{i=0}^{N} |(\eta_i)|^{p-1} \leq g_i(x) + c_i(\eta_i) \sum_{i=0}^{N} |(\eta_i)|^{p-1}$

Where $g_i(x) \equiv 0, c_i = c_i(t) = 1, hence\ a_i(x;\eta) \in CAR^*(p) for\ i = 1. consider\ when\ i = 0.$

$|a_0(x,\eta)| = \eta_0^3 \leq |\eta_0|^3 \leq \sum_{i=0}^{N} |\eta|^3 \leq g_0(x) + c_0(\eta_0) \sum_{i=0} |\eta_i|^{p-1}$

For $g_0(x) \equiv 0, c_0 = 1\ and\ p \geq 4\ therefore\ a_0(x;\eta) \in CAR^*\ (p) for\ i = 0$. Next one to show that the operator *T* is monotone i.e.

$\sum_{i}^{n} |a_0(x,\eta) - ax_i(x;\xi)|(\eta_i - \xi_i) > 0$

$[a_1(x;\eta) - ax_1(x;\xi)](\eta_1 - \xi_1) + [a_0(x;\eta) - a_0(x;\xi)](\eta_0 - \xi_0) =$

$[-(\eta_1 + \eta_0) + (\xi_1 + \xi_0)](\eta_1 - \xi_1) + (\eta_0^3 - \xi_0^3)(\eta_0 - \xi_0),$

If 2 $\eta_1\xi_1 + \eta_0\xi_1 + \eta_1\xi_0 \geq \eta^2 + \xi^2 + \eta_1\eta_0 + \xi_1\xi_0$ *and* $\eta_0\xi_0 \geq 0$ then

$[-(\eta_1 + \eta_0) + (\xi_1 + \xi_0)](\eta_1 - \xi_1) + (\eta_0 - \xi_0)^2(\eta_0^2 + \eta_0\xi_0 + \xi_0^2) \geq 0$

Hence the monotone condition is satisfied. Similarly, to show coerciveness of the operator *T*. now choose $c_1 > 0, c_2 > 0, c_3 \geq 0$ such that

$\sum_i^N a_i(x;\eta)\eta_i \geq c_1(\sum_{i=0}^n |\eta_i|^p) + c_2|\eta_0|^p - c_3$

$a_1(x;\eta)\eta_1 + a_0(x;\eta)\eta_0 \geq c_1(\sum_{i=0}^n |\eta_i|^p) + c_2|\eta_0|^p - c_3$

$-(\eta_1 + \eta_0)\eta_1 + \eta_0^4 \geq c_1(\sum_{i=0}^n |\eta_i|^p) + c_2|\eta_0|^p - c_3$

$c_3 - [-(\eta_1 + \eta_0) - \eta_0^4 - c_1(\sum_{i=0}^n |\eta_i|^p + c_2|\eta_0|^p) > 0$

$now\ choose\ c_3 > [-(\eta_1 + \eta_0) - \eta_0^4 - c_1(\sum_{i=0}^n |\eta_i|^p + c_2|\eta_0|^p)$

Then $\sum_i^N a_i(x;\eta)\eta_i \geq 0$

Clearly, the differential operator *T* is coersive . Since all the conditions of above theorem are satisfied, therefore, the differential equation has at least one weak solution.

7.0 Summary and Conclusion

Topological method which is based on the notion of some certain topological invariant is well developed method in the solution of boundary value problem of ordinary differential equation. It is base on the condition of existence and uniqueness of weak solution of boundary value problems. The study required some basic theorems from functional analysis to facilitate the understanding of the main theorem of this dissertation. The general form of differential equation

$$\sum_{i=0}^k (-1)^i \frac{d^i}{dx^i} a_i(x, u, u', \cdots, u^k) = f$$

Was used as a reference point for the study of boundary value problems of ordinary differential equation of order *2k* in this work. Other methods used by some authors are also reviewed in the work. Second order and 4th order boundary value problem with dirichlet boundary conditions were used for illustration.

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