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Vector Calculus in the Context of Space

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Abstract: *The aim of this paper is to align with the Russian school of mathematics to derive new mathematics concepts by emphasizing the physical insight of the concepts. The concept of directional derivative is derived by first building a scalar field in Euclid space, followed by defining a small change of position vector,* $\boldsymbol{ds}$ *and a unit vector,* $\boldsymbol{u}$*; the directional derivative then can be derived by using a parametric variable t. The concept of gradient can be derived from equation of directional derivative as it is simply a specific condition of directional derivative. The concept of divergence and curl is derived by first defining a new mathematics concept from the interaction of vector field with other geometric concepts lay within the space, followed by logical reasoning. Reader shall find the symbol* $\boldsymbol{\nabla}$ *denoted* $\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}$ *is naturally evolved when deriving the concepts of gradient, divergence, and curl.*

Keywords: directional derivative, gradient, divergence, curl

INTRODUCTION

The perspective that mathematics is not merely an abstract subject is a cornerstone of the Russian (and formerly Soviet) school of mathematics while Western (specifically Bourbaki-style) mathematics often emphasized formal, abstract structures. The aim of this paper in large extent is aligned with the Russian school of mathematics by building mathematical concepts/object in the Euclid space according to ‘form” and ‘equation” approach: an approach uses the “unit interval”**[1]** as object for representation by abstract equation.

LITERATURE REVIEW

Conventional vector calculus’s book always uses the abstract symbol $\boldsymbol{\nabla}$ at the very beginning to define concepts like gradient, divergence, curl. Reader has no physical insight on how the symbol $\boldsymbol{\nabla}$ and its related representation evolved. For example,

(i) Temperature gradient is denoted by ∇T
(ii) Divergence of **F** is denoted by $\nabla . \boldsymbol{F}$
(iii) Curl of **F** is denoted by $\nabla \times \mathbf{F}$
Later in the following sections, reader shall find how the symbol ∇ and its related representation evolved naturally in the study of the interaction of scalar or vector field with other geometric objects such as surface patch and curve living inside the Euclid space.

What is Field

A field is a mathematical concept lay within the concept of space: *"Concept of space is defined as the ability to locate unit interval(s)…Three-dimensional (3D) Euclidean space is defined by each point of an equilateral triangle joined to a common fourth point by using lines to form a regular tetrahedron; all six(6) lines extend the length infinitely to form an imaginary envelope of space. it could locate point, line, triangle, cube, and all shapes possess the property of volume; volume is defined as a finite envelope of 3D space."* **[2]**.

Scalar Field

The points relative to an observer within a space has physical quantity in scalar is called a scalar field.

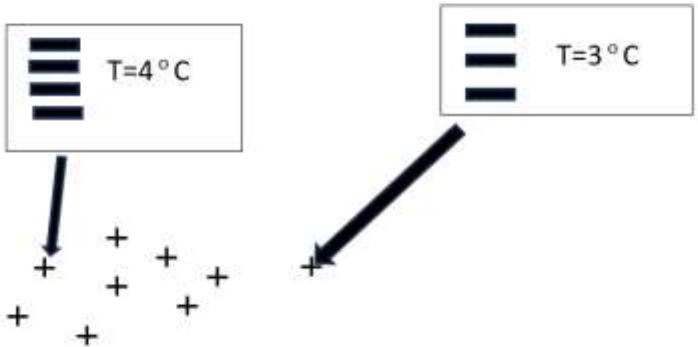


Figure 1: Temperature(scalar) field without defined coordinate system(observer)

By incorporate coordinate system, the scalar field can be represented in more abstract method. The location of points in the space are labeled by coordinate or position vector.

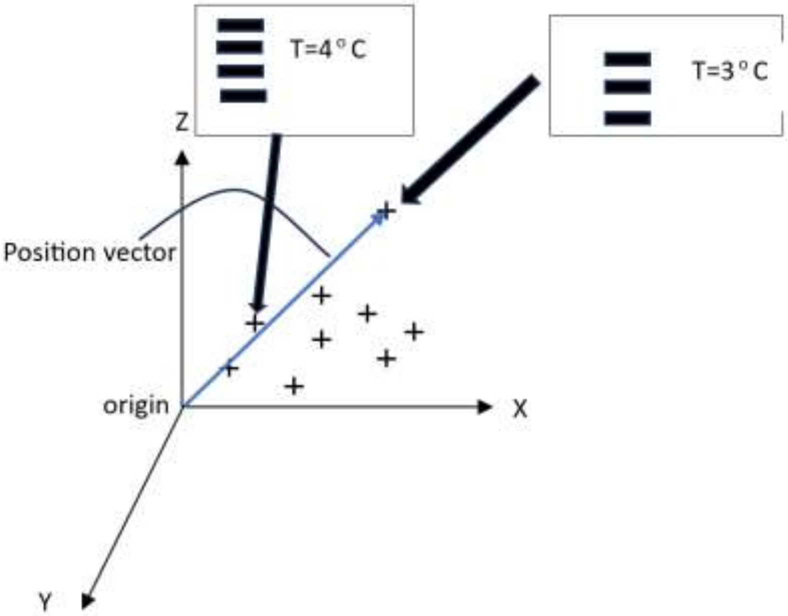


Figure 2: Temperature(scalar) field with defined coordinate system and/or position vector (observer)

From the definition of combination of vector: "*The endpoint of* $\underset{AB}{\rightarrow}$ *connects to the starting point of* $\underset{BC}{\rightarrow}$ *shall produce a new vector* $\underset{AC}{\rightarrow}$ *with a starting point at A and the endpoint at C; such reasoning is valid because the combination of* $\underset{AB}{\rightarrow}$*&* $\underset{BC}{\rightarrow}$ *is connecting the same two points of the vector* $\underset{AC}{\rightarrow}$" **[2].**

Hence, the position vector, **R** can be decomposed into component form,

$$\boldsymbol{R = xi + yj + zk}$$

In which,

i, j, k are unit vector means they are in equal length but pointed in direction orthogonal to each other.

x, y, z are magnitude of unit vector length in ***i, j, k*** direction.

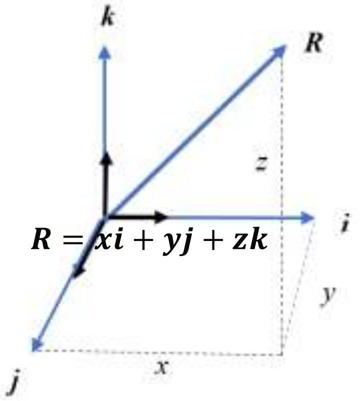


Figure 3: Position vector, **R** decomposed in components form

Vector field

A vector field has points associate to a physical quantity in vector. For example, the derivative of position vectors with time, t. is a vector field.

First, define the change of position vector,$\boldsymbol{\Delta R}$,

$$\boldsymbol{\Delta R = \Delta x i + \Delta y j + \Delta z k}$$

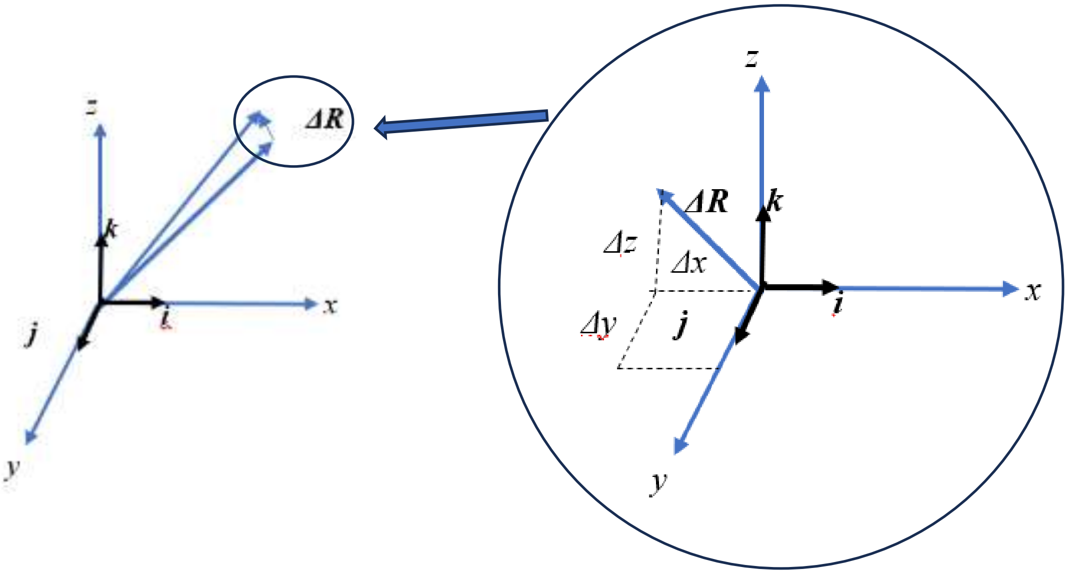


Figure 4: Change of position vector,$\boldsymbol{\Delta R}$ and its nested vector space

Take time, t as parametric variable for particle at points within the space (particle changes position as time flow), we get,

$$\boldsymbol{\frac{\Delta R}{\Delta t} = \frac{\Delta x}{\Delta t} i + \frac{\Delta y}{\Delta t} j + \frac{\Delta z}{\Delta t} k} \quad \textbf{(1)}$$

as $\boldsymbol{\Delta t}$ are equal in all direction such as $\boldsymbol{i, j, k}$ within the space.

Apply limit definition,

$$\frac{dR(t)}{dt} = \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t} i + \lim_{\Delta t \to 0} \frac{y(t+\Delta t))-y(t)}{\Delta t} j + \lim_{\Delta t \to 0} \frac{z(t+\Delta t)-z(t)}{\Delta t} k \quad (2)$$

Resultant of the rate of change of position components in direction i , j, k is the rate of change of position vector, $\frac{dR(t)}{dt}$

Representation of Velocity Field in Space

Till now, the change of position vector of points in the space which is the velocity field can be shown as below by using arrows representing the direction and magnitude of the velocity of each point.

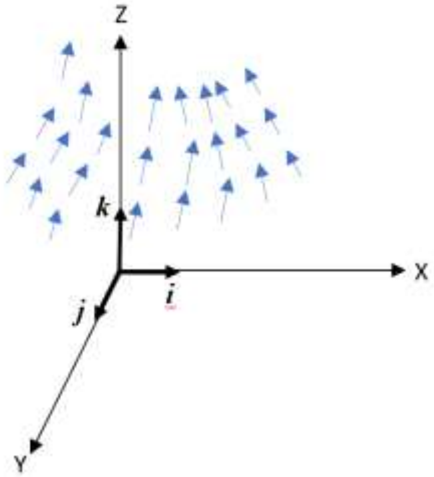
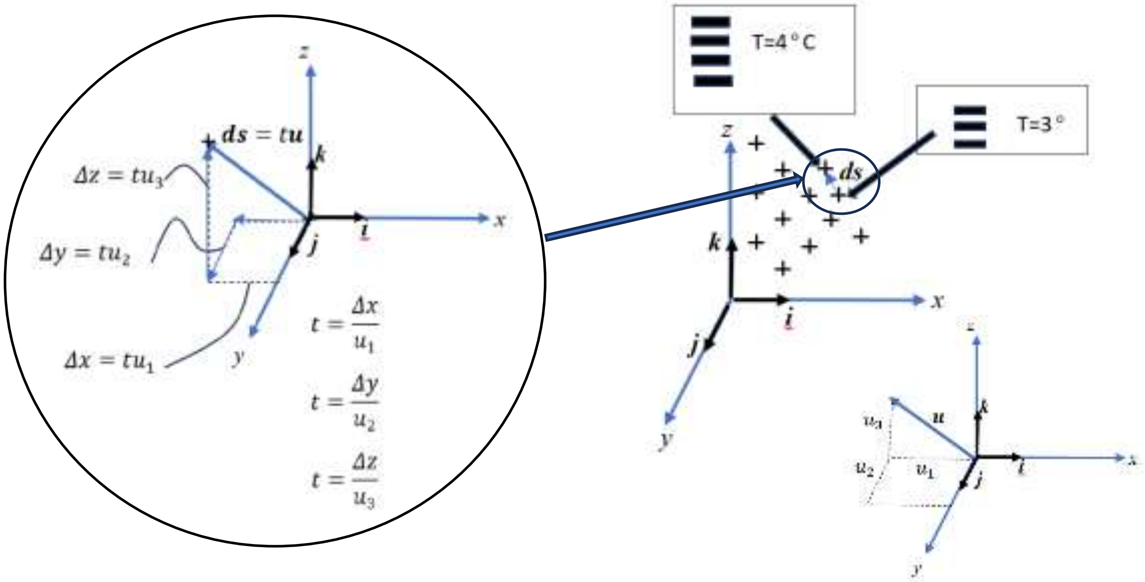


Figure 5: Velocity Field

Directional Derivative

The directional derivative, often denote as $D_s f$ measures how scalar function f changes as the position changed slightly, ds



$$\boldsymbol{ds = tu}$$
$$\boldsymbol{ds = tu_1 i + tu_2 j + tu_3 k}$$
$$\boldsymbol{\Delta s = \Delta x i + \Delta y j + \Delta z k}$$

Unit displacement vector, $\mathbf{u}$ decompose into components,
$$\boldsymbol{u = u_1 i + u_2 j + u_3 k}$$

Figure 6: Changes of temperature,T as changes of position, $\boldsymbol{ds}$

As the $\mathbf{\Delta s \neq \Delta x \neq \Delta y \neq \Delta z}$ shown in **Figure 6**. (*They are related by Pythagoras theorem:*$(\boldsymbol{\Delta s})^{\mathbf{2}} = (\boldsymbol{\Delta x})^{\mathbf{2}} + (\boldsymbol{\Delta y})^{\mathbf{2}} + (\boldsymbol{\Delta z})^{\mathbf{2}}$). Directional derivative then can differentiate provided the implementation of parametric function of *t*; as $\boldsymbol{t}$ has constant length for all direction such as $\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in space. Example :if $\boldsymbol{T}$ is function of temperature. The derivation of a composite function *T(s(t))* with respect to time, *t* is achieved using the Chain Rule in calculus. Assuming *T(s)* is a function of length, *s*; and *s* is a function of *t*, the derivative is:

$$\frac{\boldsymbol{dT(s(t))}}{\boldsymbol{dt}} = \frac{\boldsymbol{dT}}{\boldsymbol{ds}}\frac{\boldsymbol{ds}}{\boldsymbol{dt}}$$

From chain rule,

$$\boldsymbol{Directional\ derivative, D_s T} = \frac{\boldsymbol{dT}}{\boldsymbol{dt}}$$
$$= \frac{\boldsymbol{\partial T}}{\boldsymbol{\partial x}}\frac{\boldsymbol{dx}}{\boldsymbol{dt}} + \frac{\boldsymbol{\partial T}}{\boldsymbol{\partial y}}\frac{\boldsymbol{dy}}{\boldsymbol{dt}} + \frac{\boldsymbol{\partial T}}{\boldsymbol{\partial z}}\frac{\boldsymbol{dz}}{\boldsymbol{dt}} \quad (3)$$

Gradient

Gradient is defined as direction of change position that caused the greatest change in function $\boldsymbol{f}$. It can be derived from equation (3),

$$\boldsymbol{D_s T = \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}}$$

Rearrnage by apply concept of dot product of vectors,

$$\boldsymbol{\frac{dT}{dt} = \left(\frac{\partial T}{\partial x}i + \frac{\partial T}{\partial y}j + \frac{\partial T}{\partial z}k\right).\left(\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k\right)} \quad (4)$$

By denoting $\boldsymbol{\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k}$ as symbol $\boldsymbol{\nabla}$

$$\boldsymbol{\frac{dT}{dt} = \nabla T.\frac{dR}{dt}} \quad (5)$$

As dot product is maximum when angle between the two vectors is zero thus $\boldsymbol{\nabla T}$ and $\boldsymbol{\frac{dR}{dt}}$ must be in same direction when the $\boldsymbol{\frac{dT}{dt}}$ is greastest in magnitude, $\boldsymbol{\nabla T}$ is named as temperature gradient.

Divergence

New concept can be derived from the interaction of vector field with other geometry concepts lay within the space. A vector at a point of field can associated by a unit area surface, *da* by using dot product:

$$\boldsymbol{F.n}$$

In which,

F is the vector such as velocity
n is the unit normal vertor of unit surface area, *da*

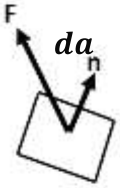


Figure 7: Vector,**F** relate to a unit area surface, *da* by using dot product:$\boldsymbol{F.n}$

In order to define vectors inward or outward a defined unit area surface, we need to define a volume or a closed surface,

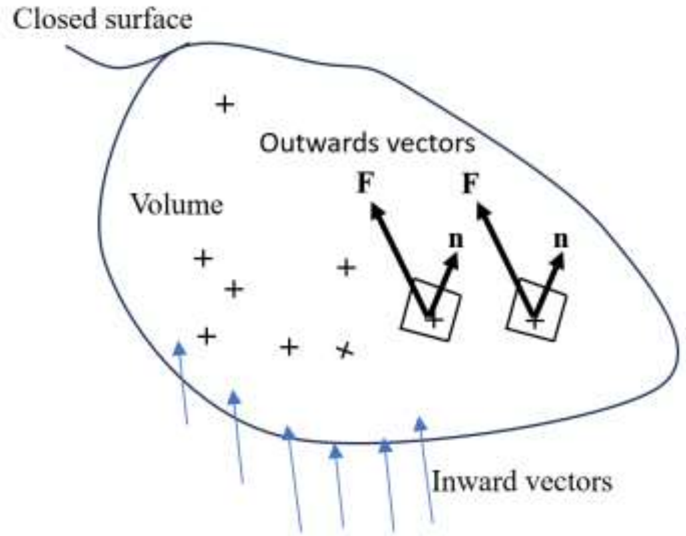


Figure 8: Concept of volume differentiate vectors as inward or outward

Outward vectors on points of closed surface can then be defined as those with arrow pointing away from the volume enclosed by surface, wherease inwards vectors can be defined as those with narrow pointing towards the said volume. The dot product between outwards vector, ***F*** and normal vector,***n*** shall labelled as positive value(+), whereas the dot product between inwards vector, ***F*** and normal vector,***n*** shall labelled as negative value(-). By assign a coordinate system or position vector to all points of the defined unit area suraface, *da*; a scalar field is derived from vectors flow through an enclosed surface. The derived scalar field has points defined on the said enclosed surface.

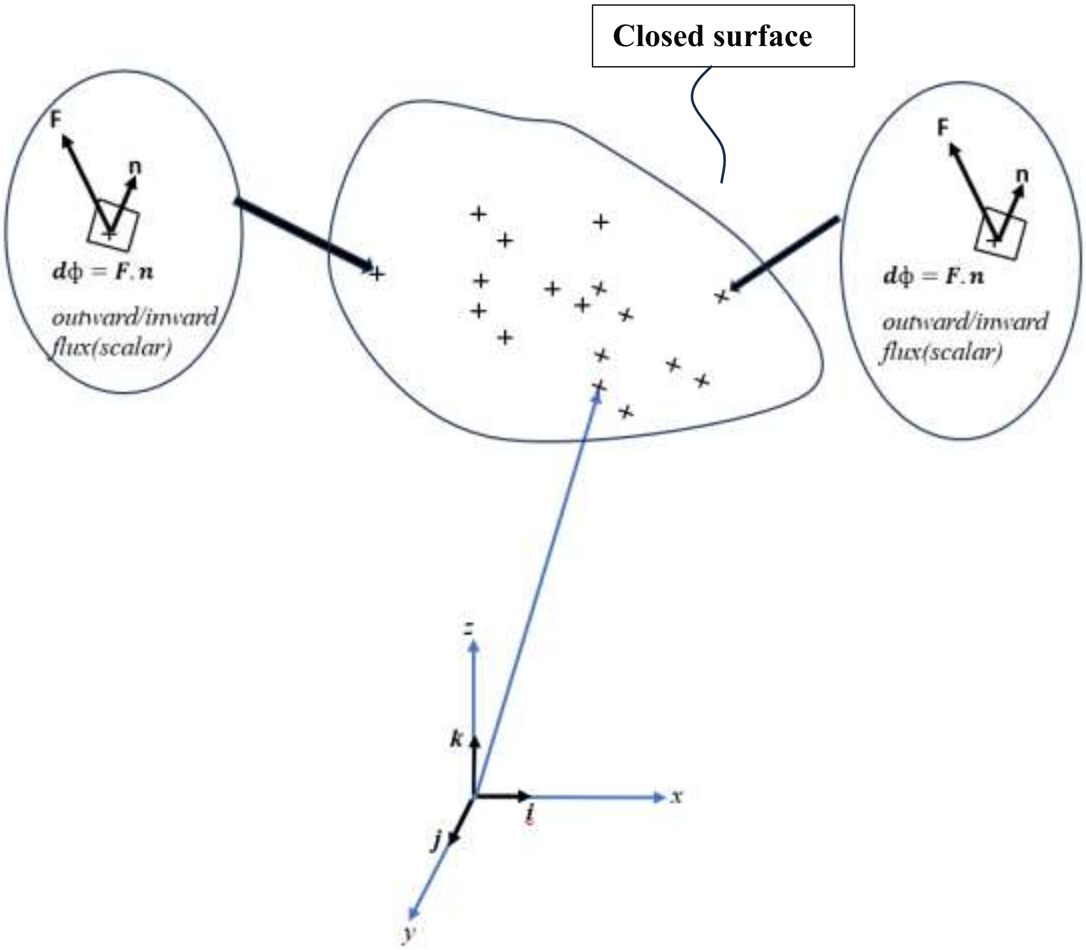


Figure 9: A scalar field, ϕ with points defined on the enclosed surface

Flux calculated for each point can be shown in 3D graph. Let $\boldsymbol{z}(\boldsymbol{x},\boldsymbol{y})$ is the flux of unit area surface, *da* of each point on the enclosed surface, the domain is the points lay within the projection of enclosed surface on the x-y plane.

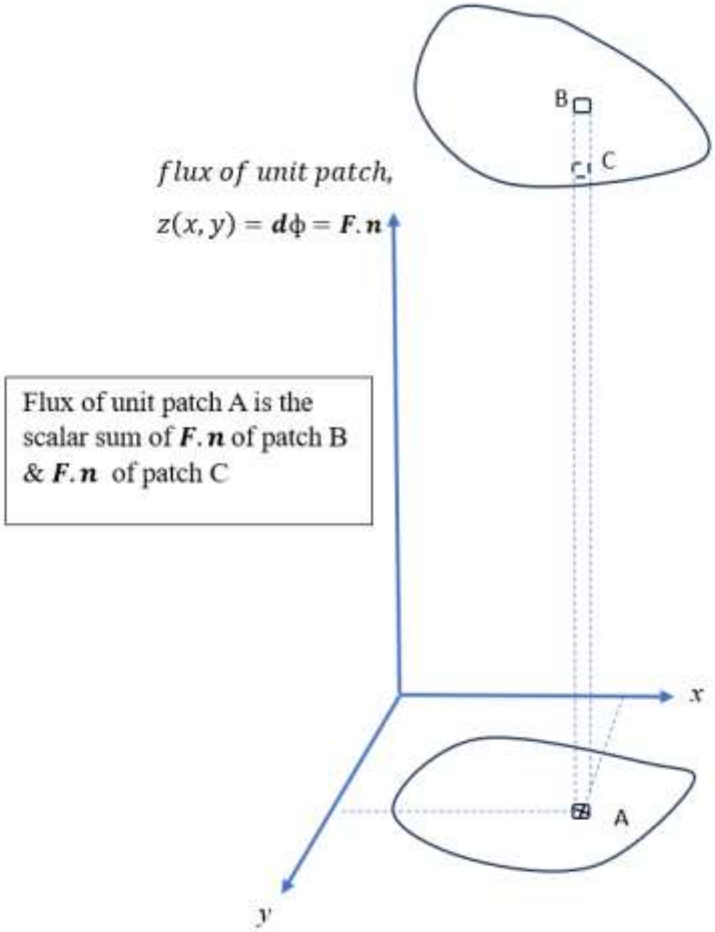


Figure 10: 3D graph of flux, $\boldsymbol{F.n}$

Then the total flux through the whole enclosed surface, is the Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} g(x_i, y_j)\, da(x_i, y_j)$
It shall be calculate by surface integral.

Volume of each point=$\boldsymbol{F.nda}$

Total volume,V=$\iint \boldsymbol{F.n\,da} \quad (6)$

Of course, instead of 3D graph, the sum of flux through the enclosed surface can be calculated by volume integral. By referring to **Figure 8**, the volume of the enclosed surface can be divided into multiple unit volumes, each unit volume labeled by a point in coordinate x,y,z. Each unit volume then can be calculated the $d\phi = \boldsymbol{F.n}$

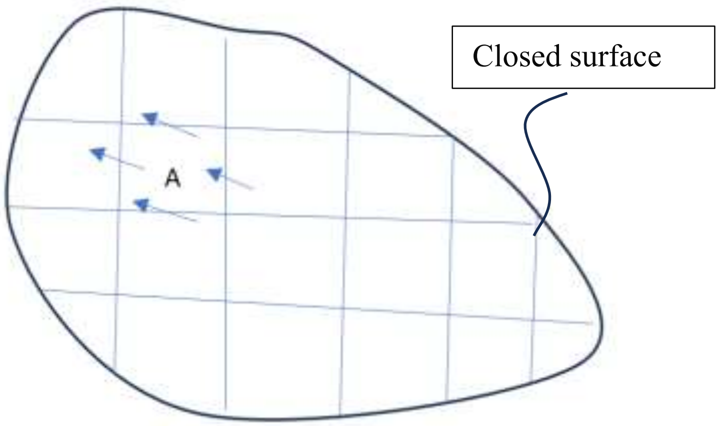


Figure 11: Volume of enclosed surface divided into multiple small volumes

As from **Figure 11**, the unit volume A has all surfaces share with other unit volumes. The $\boldsymbol{d}\phi = \boldsymbol{F}.\boldsymbol{n}$ for all surfaces of unit volume A shall be cancelled out while summing the $\boldsymbol{d}\phi$ with all adjacent unit volumes. It is obvious that the net sum of $\boldsymbol{d}\phi$ of all unit volumes shall be the sum of $\boldsymbol{d}\phi$ of the enclosed surface of the whole body because this is the outer surface which don't share with other unit volumes.

The total flux through the whole enclosed surface, is the Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{p} g(x_i, y_j, z_k)\,\Delta v$

In which,

$g(x, y, z)$ is $\boldsymbol{d}\phi$ for the unit volume

$dv(x, y, z)$ is the unit volume

It shall be calculate by volume integral for net sum of flux through all the unit volumes,

$$\iiint d\phi \;\, dv \;(7)$$

From above, the calculated volume integral of body shown in (7) shall equal to surface integral of the whole enclosed surface of the body shown in (6).

$$\iiint d\phi \;\, dv = \iint F.n\, da \quad (8)$$

$\boldsymbol{d}\phi$ for a unit volume can be wrtten in diffferentiatial form,

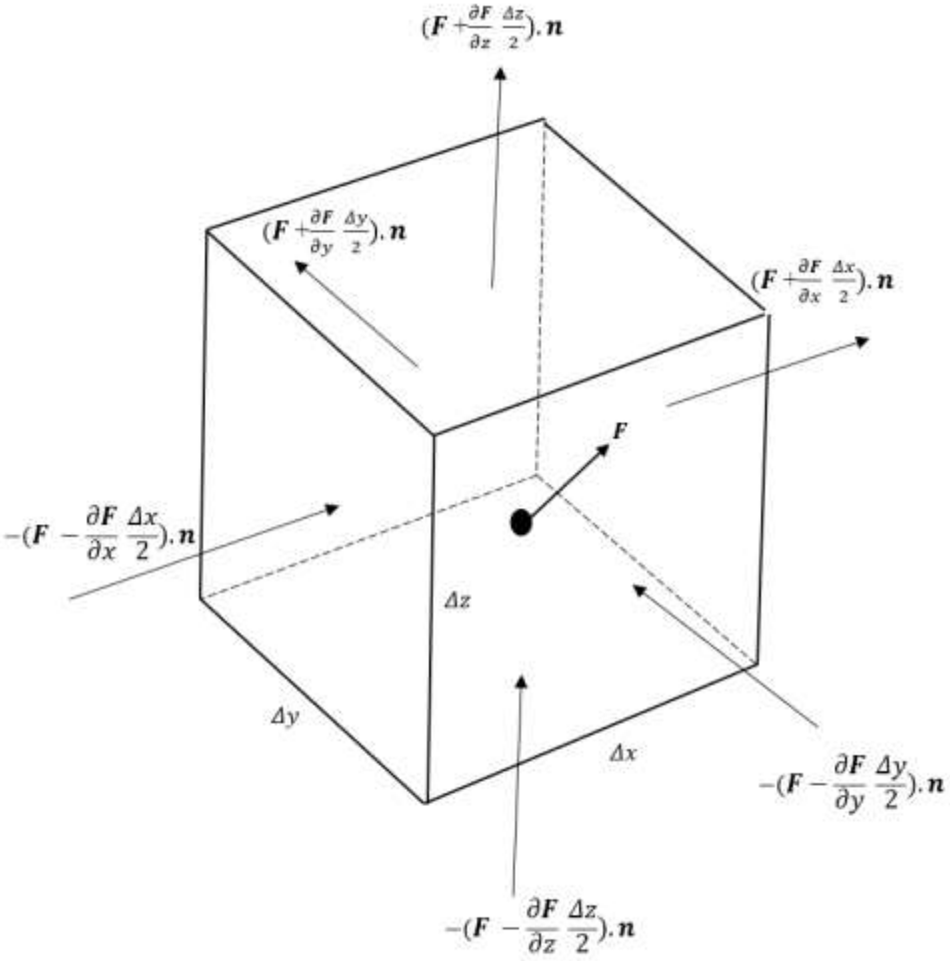


Figure 12: Flux through a unit volume

Flux through surfaces $\Delta y\Delta z$ at x direction,

$$d\phi_x = -(F - \frac{\partial F}{\partial x}\frac{\Delta x}{2}).n\,\Delta y\Delta z + (F + \frac{\partial F}{\partial x}\frac{\Delta x}{2}).n\,\Delta y\Delta z$$

$$= \frac{\partial F}{\partial x}.n\Delta x\Delta y\Delta z$$

$$= \frac{\partial F_x}{\partial x}\Delta x\Delta y\Delta z$$

By label $F.n$ as F_x due to the normal vector is at x direction.

Flux through surfaces $\Delta x\Delta z$ at y direction,

$$d\phi_y = -(F - \frac{\partial F}{\partial y}\frac{\Delta y}{2}).n\,\Delta x\Delta z + (F + \frac{\partial F}{\partial y}\frac{\Delta y}{2}).n\,\Delta x\Delta z$$

$$= \frac{\partial F}{\partial y}.n\Delta x\Delta y\Delta z$$

$$= \frac{\partial F_y}{\partial y}\Delta x\Delta y\Delta z$$

By label $\boldsymbol{F.n}$ as $\boldsymbol{F_y}$ due to the normal vector is at y direction.
Flux through surfaces $\boldsymbol{\Delta x \Delta y}$ at y direction,

$$d\phi_z = -\left(F - \frac{\partial F}{\partial z}\frac{\Delta z}{2}\right).n\,\Delta x\Delta y + \left(F + \frac{\partial F}{\partial z}\frac{\Delta z}{2}\right).n\,\Delta x\Delta y$$
$$= \frac{\partial F}{\partial z}.n\Delta x\Delta y\Delta z$$
$$= \frac{\partial F_z}{\partial z}\Delta v$$

By label $\boldsymbol{F.n}$ as $\boldsymbol{F_z}$ due to the normal vector is at z direction.
Net flux through unit volume,

$$d\phi = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x\Delta y\Delta z \quad (10)$$

As $\boldsymbol{F_x}, \boldsymbol{F_y}, \boldsymbol{F_z}$ can be treated as components of a vector $\boldsymbol{F}$,

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x\Delta y\Delta z = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(F_x i + F_y j + F_z k)\,\Delta x\Delta y\Delta z$$
$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).F\,(\Delta x\Delta y\Delta z) \quad (11)$$
$$= \nabla.F\,(\Delta x\Delta y\Delta z) \quad (12)$$
$$d\phi = \mathrm{div}\,F\Delta v$$

In which $\boldsymbol{\nabla.F}$ **named as divergence of** $\boldsymbol{F, div\,F}$
$\boldsymbol{\Delta v}$ is the unit volume
Then equation (8) can be written as

$$\iiint \mathrm{div}\,F\;dv = \iint F.n\,da \quad (13)$$

Equation (13) is Divergence Theorem, also known as Gauss's Theorem**.**

Curl

Other new concept can be derived from the interaction of vector field with other geometry concepts lay within the space, this time interact with a curve. A vector at a point of field can accosiate with a line segment, $\boldsymbol{ds}$ of a curve by using dot product:

$\boldsymbol{F.ds}$ (a scalar)

In which,
$\boldsymbol{F}$= Any vector
$\boldsymbol{ds}$ = a segment of curve

Figure 13: $\boldsymbol{F.ds}$ (a scalar)

In order to see the circulation characteristics of the vector field, the above specified curve must be in closed loop. The clockwise direction of tangential component of $\boldsymbol{F}$ is labeled as negative (-) whereas the anticlockwise direction is labeled as positive (+).

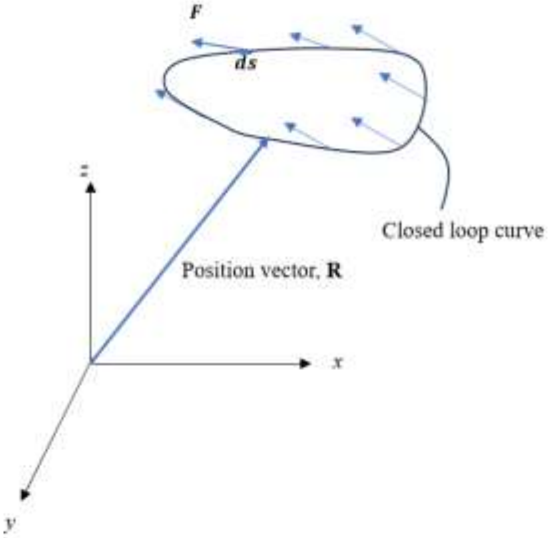


Figure 14: Circulation of vector in a loop is equal to sum of $\boldsymbol{F.ds}$ of the curve.

The net sum of $\boldsymbol{F.ds}$ around the curve is the Riemann sum $\sum_{i=1}^{m} \boldsymbol{g(s_i)\,\Delta s}$ in which $\boldsymbol{g(s)}$ is the magnitude of tangential component of $\boldsymbol{F}$.
In 2D graph, let $\boldsymbol{g(s)}$ and $\boldsymbol{\Delta s}$ are parametric by variable t,

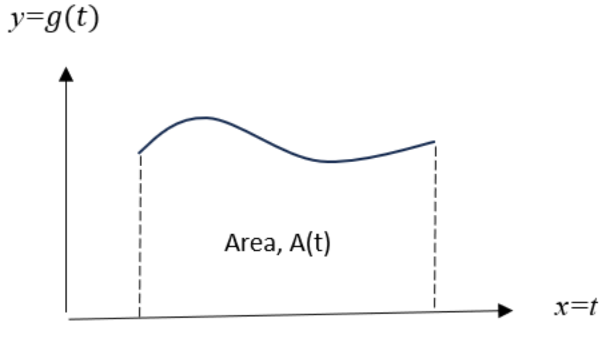


Figure 15: Graph of $\boldsymbol{g(s)}$ versus t

Then net sum of $\boldsymbol{F.ds}$ around the curve is the area under the graph,

$$\boldsymbol{A' = g(t)}$$

$$\boldsymbol{A = \int g(t)\ dt}$$

Or in dot product of vectors convention,

$$\textit{Net sum of circulation of F around the curve} = \oint \boldsymbol{F}.\boldsymbol{ds} \quad (\mathbf{14})$$

Of course, instead of 2D graph, then net sum of $\boldsymbol{F}.\boldsymbol{ds}$ around the curve can be calculated by surface integral. By referring to **Figure 14**, the curve can be a boundary curve of a open surface in any form; the surface then can be divided into multiple unit patches of surface, each boundary curve of a unit patch labeled by coordinate x,y,z. Each unit patch then can be calculated the $\boldsymbol{F}.\boldsymbol{ds}$ around its boundary curve**.**

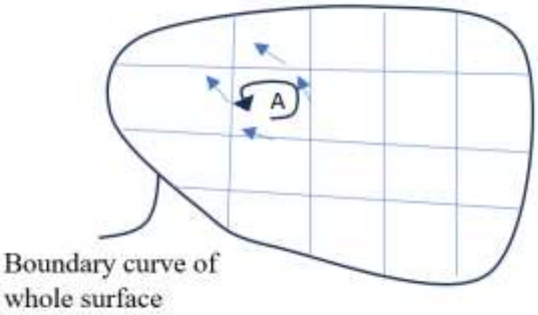


Figure 16: Open surface of a closed boundary curve divided into multiple small patches.

As from **Figure 16**, the unit patch, A has all boundary curves share with other unit patches. The $\boldsymbol{F}.\boldsymbol{ds}$ for the closed curve of unit patch, A shall be cancelled out while summing up with the $\boldsymbol{F}.\boldsymbol{ds}$ of all adjacent unit patches as the sign of $\boldsymbol{F}.\boldsymbol{ds}$ are opposite between two neighboring unit patches. It is obvious that the net sum of $\boldsymbol{F}.\boldsymbol{ds}$ of all unit patches is equal to $\boldsymbol{F}.\boldsymbol{ds}$ around the boundary curve of the open surface.

Let’s see the circulation of a $\boldsymbol{F}$ around a loop of an infinitesimal small unit patch,

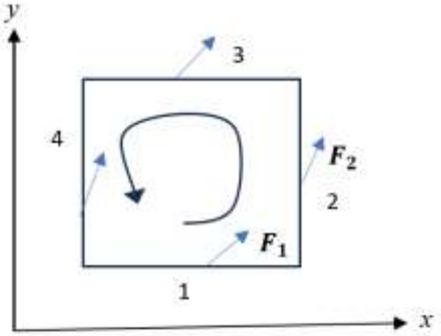


Figure 17: Line integral of an infinitesimal unit patch

The $\boldsymbol{F}.\boldsymbol{ds}$ or unit interval (1) & (3),

$$\boldsymbol{F_t(1).\Delta x - \left(F_t(1) + \frac{\partial F_t(1)}{\partial y}\Delta y\right)\Delta x = -\frac{\partial F_t(1)}{\partial y}\Delta y\Delta x}$$

In which $\boldsymbol{F_t(1)}$ is the magnitude of tangential component of vector $\boldsymbol{F_1}$
The $\boldsymbol{F.ds}$ of unit interval (2) & (4),

$$\boldsymbol{F_t\,(2).\Delta y - \left(F_t(2) - \frac{\partial F_t(2)}{\partial x}\Delta x\right)\Delta y = \frac{\partial F_t(2)}{\partial x}\Delta x\Delta y}$$

In which $\boldsymbol{F_t(2)}$ is the magnitude of tangential component of vector $\boldsymbol{F_2}$
The $\boldsymbol{F.ds}$ around the loop,

$$\oint \boldsymbol{F.ds} = (\frac{\boldsymbol{\partial F_t(2)}}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial F_t(1)}}{\boldsymbol{\partial y}})\boldsymbol{\Delta x\Delta y}$$

$$= (\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\boldsymbol{i} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\boldsymbol{j}\ \right) \times \boldsymbol{F})\ \ (\boldsymbol{\Delta x\Delta y})$$

As $\boldsymbol{F} = \boldsymbol{F_t(1)i} + \boldsymbol{F_t(2)j}$

$$= (\boldsymbol{\nabla} \times \boldsymbol{F}\,)(\boldsymbol{\Delta x\Delta y})$$

$\boldsymbol{\nabla} \times \boldsymbol{F}$ is defined as curl of vector field,

Line integral of the unit patch, $\oint \boldsymbol{F.ds} = (\boldsymbol{\nabla} \times \boldsymbol{F}\,).\boldsymbol{nda} \quad \mathbf{(15)}$

In which $(\boldsymbol{\nabla} \times \boldsymbol{F}\,)$ is always in the normal direction to *da,* hence the angle of vector $(\boldsymbol{\nabla} \times \boldsymbol{F}\,)$ with **n** is zero.

The sum of $\boldsymbol{F.ds}$ for all unit patches of open surface then can calculated by volume under surface in the 3D graph similar to **Figure 10**.

$$\boldsymbol{Total\ volume, V} = \iint (\boldsymbol{\nabla} \times \boldsymbol{F}\,).\boldsymbol{nda} \quad \mathbf{(16)}$$

As net sum of $\boldsymbol{F.ds}$ of all unit patches is equal to $\boldsymbol{F.ds}$ around the boundary curve of the open surface. Hence,

$$\boldsymbol{Line\ integral\ around\ boundary\ curve,} \oint \boldsymbol{F.ds} = \iint (\boldsymbol{\nabla} \times \boldsymbol{F}\,).\boldsymbol{n\,da} \quad \mathbf{(17)}$$

Equation (17) is Stokes Theorem.

Conclusion

The new mathematical concepts can be derived from the interactions of geometric object with the scalar or vector field in the Euclid space. The symbol $\boldsymbol{\nabla}$ denoted $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\boldsymbol{i} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\boldsymbol{j} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}\boldsymbol{k}$ is naturally evolved when deriving concept of gradient, divergence and curl; these concepts are closely related to their physical meaning through out the derivation process. Hence, mathematics is not merely an abstract subject according to the approach utilised in this paper.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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