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Face Antimagic Labeling of House Graphs, Double House Graphs, and Triangular Ladder Graphs

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Abstract: *This study investigates the $(\boldsymbol{a}, \boldsymbol{d})$-face antimagic labeling on three classes of planar graphs, namely the house graph, the double house graph, and the triangular ladder graph. An $(\boldsymbol{a}, \boldsymbol{d})$-face antimagic labeling is defined as a bijective function from the set of vertices, edges, and faces to the set of positive integers, such that the sum of the labels assigned to the vertices and edges incident with each face forms an arithmetic sequence. In this work, specific labeling constructions are developed for the vertices, edges, and faces of each graph to ensure that the face antimagic property is satisfied. Illustrative examples are presented to demonstrate the labeling schemes that produce face weights forming an arithmetic sequence with a fixed common difference of $\boldsymbol{d} = \mathbf{1}$. The findings confirm that all three classes of graphs admit valid face antimagic labelings. This research contributes to the advancement of graph labeling theory, particularly in the study of structured planar graphs.*

Keywords: Graph labeling, *face antimagic*, house graph, double house graph, triangular ladder graph

INTRODUCTION

Graph theory encompasses numerous topics for study, one of which is graph labeling. Graph labeling is the assignment of integers to the vertices, edges, or faces of a graph with certain conditions (Roopa, 2019). The concept of graph labeling was introduced by Rosa (1967) in the context of vertex labeling. Various types of graph labeling have been introduced, one of which is magic and antimagic labeling.

Magic labeling was introduced by Kotzig and Rosa (1970) as a bijection $f: V \cup E \rightarrow \{1,2, \dots, |V| + |E|\}$, where V and E denote the sets of vertices and edges of a graph, respectively, such that for all edges ab, $f(a) + f(b) + f(ab)$ is a constant (Gallian, 2024). This constant value is referred to as the magic constant. Various applications of graph labeling include problems in graph decomposition, radar pulse code design, X-ray crystallography, and modeling communication networks (Kotzig, 1970).

Graph labeling has developed into various types, including vertex magic labeling, edge magic labeling, total magic labeling, vertex antimagic labeling, edge antimagic labeling, total face antimagic labeling, face magic labeling, and face antimagic labeling (Siddiqui, 2013). Let $G\ (V, E, F)$ be a finite connected planar graph with no loops or multiple edges, where $V(G), E(G)$, and $F(G)$ represent the sets of vertices, edges, and faces, respectively. A face antimagic labeling of type $(1,1,1)$ assigns a label from the set $\{1,2, \ldots, |\ V(G)\ | + |\ E(G)\ | + |\ F(G)\ |\}$ to the vertices, edges, and faces of a graph G such that each vertex, edge, and face receives exactly one label and each number is used exactly once as a label (Siddiqui, 2013).

Face antimagic labeling has been widely discussed and applied to various types of graphs. For example, Siddiqui (2013) and Ahmed (2017) discuss d-antimagic labeling of type $(1, 1, 1)$. This type of labeling combines a type $(1,1,1)$ labeling, which assigns unique labels to vertices, edges, and faces, and d-antimagic labeling, an extension of antimagic labeling introduced by Baca and Miller (2003). Ahmed and Babujee (2017) investigated this labeling for $d = 0, 1, 2$ on the strong face forms of wheel graphs, ladder graphs, and friendship graphs, as well as their generalizations. Meanwhile, Siddiqui et al. (2013) discussed the existence of this labeling for Jahangir graphs. Another study by Kuppan and Shobana (2023) discussed face antimagic labeling for double duplication forms of ladder graphs, tadpole graphs, and path graphs that are duplicated m times.

This research studies face antimagic labeling for house graphs, double house graphs, and triangular ladder graphs. The objective of the research is to formulate and prove theorems concerning face antimagic labeling for house graphs, double house graphs, and triangular ladder graphs, which are developed based on patterns obtained through graph structure analysis. The graphs used in this research are planar derivatives of ladder structures with distinctive combinatorial characteristics.

METHODOLOGY

The research conducted is a literature study aimed at generating theorems on face antimagic labelling for house graphs, double house graphs, and triangular ladder graphs. The initial stage involves collecting literature that will be used in this research. This literature study includes learning about the types of graphs used in the thesis and the labelling applied. The graph labelings studied include magic, antimagic, face magic, and face antimagic. Meanwhile, the graphs used are house graphs, double house graphs, and triangular ladder graphs.

The next stage is constructing these three graphs based on their respective definitions. Then, for each graph, a step-by-step labeling process is carried out, starting with small-order graphs, those with the smallest possible number of edges, vertices, and faces, and then proceeding to larger-order graphs. The final stage is formulating theorems based on the results of the labeling observations.

DISCUSSION

Face Antimagic Labeling on House Graphs

Before discussing face antimagic labeling on house graphs, it is important to understand the basic definition of a house graph.

Definition 3.1 (Kang, 2016) *A house graph (H_n) is a graph formed from a ladder graph (L_n) by adding n vertices c_i and connecting c_i to vertices b_i and b_{i+1}.*

Example 3.1. Figure 3.1. shows a house graph for $n = 2$.

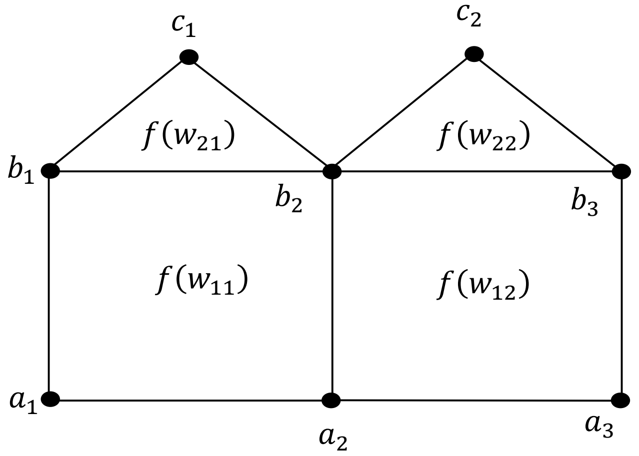

Figure 3.1 Graph H_2

On graph H_2, the vertex, edge, and face sets are, in order,
$V(H_2) = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}$
$E(H_2) = \{a_1b_1, a_2b_2, a_3b_3, b_1c_1, b_2c_1, b_2c_2, b_3c_2, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$
$\mathcal{F}(H_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22})\}$
After discussing house graphs, the definition of planar graphs is given in the following Definition 3.2.

Definition 3.2 (Deo, 2016) *A graph is called a planar graph if the graph can be drawn on a plane without the edges of the graph intersecting.*
After discussing planar graphs, the definitions of a vertex, edge, and face are given in the following Definition 3.3.

Definition 3.3 (Deo, 2016) *A vertex is the main element of a graph where edges meet. An edge is a connection between two vertices in a graph. A face is a region bounded by the edges of a graph when the graph is drawn on a plane.*
After discussing vertices, edges, and faces, the definition of face-antimagic labeling is given in the following Definition 3.4.

Definition 3.4 (Bača, 2001) *A connected planar graph $G(V, E, F)$ is said to be (a, d)-face antimagic if there exist positive integers $a, d \in N$ and a bijective mapping $\delta: E(G) \to \{1,2, \ldots, |E(G)|\}$ such that*

the induced mapping $\delta^*: F(G) \to W$ is also a bijective mapping, where $W = \{w(f): f \in F(G)\} = \{a, a+d, a+2d, \dots, a+(|F(G)|-1)d\}$ is the set of weights of the faces. If $G(V,E,F)$ is an (a,d)-face antimagic graph and $\delta: E(G) \to \{1,2,\dots,|E(G)|\}$ is a corresponding bijective mapping of G, then δ is called an (a,d)-face antimagic labeling of G.
One of the variations of face antimagic labeling is (a,d)-face antimagic labeling of type $(1,1,1)$. This labeling is given by Definition 3.5.

Definition 3.5 (Bača, 2017) *A type $(1,1,1)$ labeling assigns labels from the set $\{1,2,\dots,p+q+f\}$ to the vertices, edges, and faces of a plane graph G such that each vertex, edge, and face receives exactly one label, and each number is used exactly once as a label.*
Based on the definition of face antimagic labeling on house graphs, the following Theorem 3.1 is obtained.

Theorem 3.1 *The house graph H_n is an (a,d)-face antimagic graph.*
Proof. We first construct the labeling functions for the vertices, edges, and faces. Let $g: V(H_n) \to \{1,2,\dots n+1\}$ and $n = 1,2,3,4,\dots$. We define the labeling for the vertices, edges, and faces of the house graph H_n as follows.
Vertex Labeling:
$f(a_i) = i$, for $1 \le i \le n-1$
$f(b_i) = 2n - i + 3$, for $1 \le i \le n-1$
$f(c_i) = 2n + i + 2$, for $1 \le i \le n-1$
with $i = 1,2,\dots n+1$
Edge Labeling:
$f(a_i b_i) = 3n + i + 2$, for $1 \le i \le n-1$
$f(b_i c_i) = 4n + i + 3$, for $1 \le i \le n-1$
$f(a_i a_{i+1}) = 6n - i + 4$, for $1 \le i \le n-1$
$f(b_{i+1} c_i) = 6n + i + 3$, for $1 \le i \le n-1$
$f(b_i b_{i+1}) = 8n - i + 4$, for $1 \le i \le n-1$
with $i = 1,2,\dots n+1$
Face Labeling:
$f(w_{1i}) = 8n + 2i + 2$, for $1 \le i \le n-1$
$f(w_{2i}) = 8n + 2i + 3$, for $1 \le i \le n-1$
with $i = 1,2,\dots n+1$
Based on the vertex, edge, and face labeling, the face weights are obtained as follows.

$$\begin{aligned} W_{1i} &= f(a_i) + f(b_i) + f(a_i b_i) + f(a_i a_{i+1}) + f(b_i b_{i+1}) + f(w_{1i}) \\ &= i + 2n - i + 3 + 3n + i + 2 + 6n - i + 4 + 8n - i + 4 + 8n + 2i + 2 \\ &= 2n + 3n + 6n + 8n + 8n + i - i + i - i - i + 2i + 3 + 2 + 4 + 4 + 2 \\ &= 27n + i + 15 \end{aligned}$$

$$\begin{aligned} W_{2i} &= f(b_i) + f(c_i) + f(b_i c_i) + f(b_{i+1} c_i) + f(b_i b_{i+1}) + f(w_{2i}) \\ &= 2n - i + 3 + 2n + i + 2 + 4n + i + 3 + 6n + i + 3 + 8n - i + 4 + 8n + 2i + 3 \\ &= 2n + 2n + 4n + 6n + 8n + 8n - i + i + i + i - i + 2i + +3 + 2 + 3 + 3 + 4 + 3 \\ &= 30n + 3i + 18 \end{aligned}$$

Thus, the weight of each face is different. Hence, the graph H_n admits a (a,d)-face antimagic labeling and the theorem is proven.

Example 3.2 Face antimagic labeling on a house graph H_2 with $V(DH_2) = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}$, $E(DH_2) = \{a_1b_1, a_2b_2, a_3b_3, b_1c_1, b_2c_1, b_2c_2, b_3c_2, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$, and $\mathcal{F}(DH_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22})\}$. Then, we define the vertex, edge, and face labeling for the graph using the labeling pattern that has been obtained.

Vertex labeling:
$f(a_1) = 1, f(a_2) = 2, f(a_3) = 3$
$f(b_1) = 6, f(b_2) = 5, f(b_3) = 4$
$f(c_1) = 7, f(c_2) = 8$

Edge labeling:
$f(a_1b_1) = 9, f(a_2b_2) = 10, f(a_3b_3) = 11$
$f(b_1c_1) = 12, f(b_2c_2) = 13$
$f(a_1a_{1+1}) = 15, f(a_2a_{2+1}) = 14$
$f(b_{1+1}c_1) = 16, f(b_{2+1}c_2) = 17$
$f(b_1b_{1+1}) = 19, f(b_2b_{2+1}) = 18$

Face labeling:
$f(w_{11}) = 20, f(w_{12}) = 22$
$f(w_{21}) = 21, f(w_{22}) = 23$

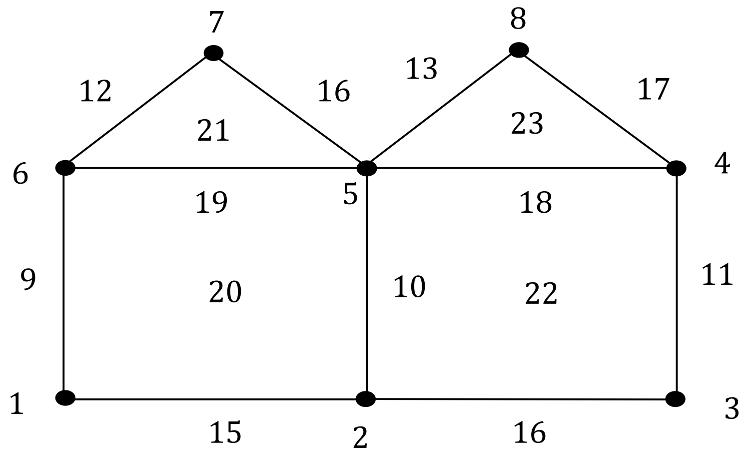

Figure 3.2 Face antimagic labeling on the H_2 graph

Figure 3.2 illustrates Theorem 3.1 applied to *the* H_2 graph. Given a house graph with $n = 2$ and $i = 1,2,3$.

Subsequently, the values of $W_{11}, W_{21}, W_{12}, and\ W_{22}$ are obtained as follows.

$W_{11} = 14 + 53 + 20 = 87$
$W_{12} = 14 + 53 + 22 = 89$
$W_{21} = 18 + 47 + 21 = 86$
$W_{22} = 17 + 48 + 23 = 88$

Based on the results obtained, the set of weights W on the graph resulting from the H_2 operation is as follows:

$$W = \{86,87,88,89\}$$

The set W forms an arithmetic sequence with a common difference $d = 1$, so the labeling is called an $\{86, 87, 88, 89\}$-face antimagic labeling.

Face Antimagic Labeling on Double House Graph

As in the previous section, it is important to understand the basic definition of a graph used in this section. For this section, the basic definition of a double house graph is given as follows.

Definition 3.6 (Sudhahar, 2022) *A double house graph (DH_n) is a graph formed from a ladder graph (Ln) by adding n vertices c_i and connecting c_i to vertices b_i and b_{i+1}, and also adding n vertices wi and connecting d_i to vertices a_i and a_{i+1}.*

Example 3.3. Figure 3.3 illustrates the double house graph DH_n for $n = 2$.

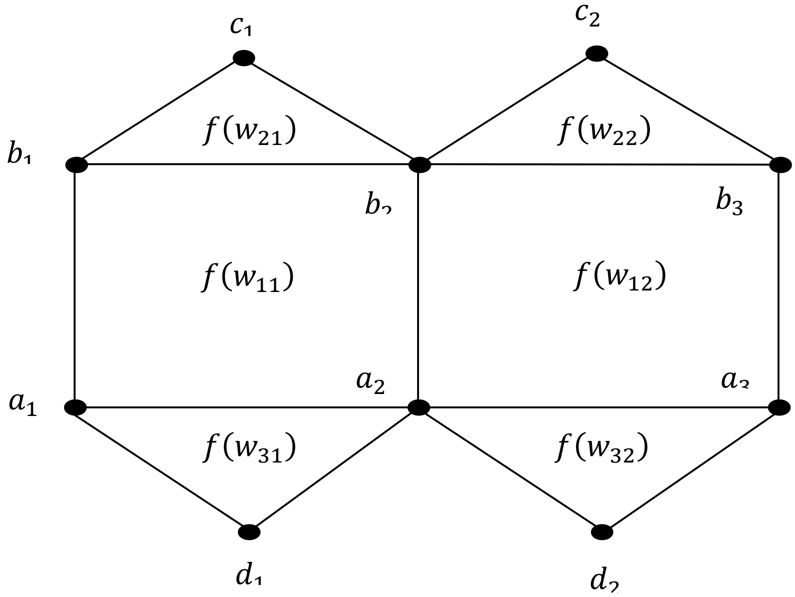

Figure 3.3 Graph DH_2

For this graph,

$V(DH_2) = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2\}$

$E(DH_2) = \{a_1b_1, a_2b_2, a_3b_3, b_1c_1, b_2c_1, b_2c_2, b_3c_2, a_1d_1, a_2d_1, a_2d_2, a_3d_2, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$

$\mathcal{F}(DH_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22}), f(w_{31}), f(w_{32})\}$

are vertex, edge, and face sets, respectively.

Based on the definition of face antimagic labeling on house graphs, the following Theorem 3.2 is obtained.

Theorem 3.2 *The double house graph DH_n is an (a, d)-face antimagic graph.*

Proof: We first construct the labeling functions for the vertices, edges, and faces. Let $g: V(\mathrm{DH_n}) \rightarrow \{1,2, \dots\ n+1\}$ and $n = 1,2,3,4, \dots$. We define the labeling for the vertices, edges, and faces of the double house graph $\mathrm{DH_n}$ as follows.

Vertex labelling:

$f(a_i) = i$, for $1 \leq i \leq n-1$

$f(b_i) = 2n - i + 3$, for $1 \leq i \leq n-1$

$f(c_i) = 2n + i + 2$, for $1 \leq i \leq n-1$

$f(d_i) = 4n - i + 3$, for $1 \leq i \leq n-1$

With $i = 1,2, \dots\ n+1$

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Edge labelling:
$f(a_ib_i) = 4n + i + 2$, for $1 \leq i \leq n - 1$
$f(b_ic_i) = 5n + i + 3$, for $1 \leq i \leq n - 1$
$f(a_id_i) = 7n - i + 4$, for $1 \leq i \leq n - 1$
$f(a_ia_{i+1}) = 8n - i + 4$, for $1 \leq i \leq n - 1$
$f(b_ib_{i+1}) = 9n - i + 4$, for $1 \leq i \leq n - 1$
$f(b_{i+1}c_i) = 9n + i + 3$, for $1 \leq i \leq n - 1$
$f(a_id_{i+1}) = 10n + i + 3$, for $1 \leq i \leq n - 1$
With $i = 1,2, \dots n + 1$
Face labelling:
$f(w_{1i}) = 11n + 3i + 1$, for $1 \leq i \leq n - 1$
$f(w_{2i}) = 11n + 3i + 2$, for $1 \leq i \leq n - 1$
$f(w_{3i}) = 11n + 3i + 3$, for $1 \leq i \leq n - 1$
With $i = 1,2, \dots n + 1, n = 1,2,3,4, \dots$

Based on the vertex, edge, and face labeling, the face weights are obtained as follows.

$$
\begin{aligned}
W_{1i} &= f(a_1) + f(b_i) + f(a_ib_i) + f(a_ia_{i+1}) + f(b_ib_{i+1}) + f(w_{1i}) \\
&= i + 2n - i + 3 + 4n + i + 2 + 8n - i + 4 + 9n - i + 4 + 11n + 3i + 1 \\
&= 2n + 4n + 8n + 9n + 11n + i - i + i - i - i + 3i + 3 + 2 + 4 + 4 + 1 \\
&= 34n + 2i + 14 \\
W_{2i} &= f(b_1) + f(c_i) + f(b_ic_i) + f(b_{i+1}c_i) + f(b_ib_{i+1}) + f(w_{2i}) \\
&= 2n - i + 3 + 2n + i + 2 + 5n + i + 3 + 9n + i + 3 + 9n - i + 4 + 11n + 3i + 2 \\
&= 2n + 2n + 5n + 9n + 9n + 11n - i + i + i + i - i + 3i + 3 + 2 + 3 + 3 + 4 + 2 \\
&= 38n + 4i + 17 \\
W_{3i} &= f(a_1) + f(d_i) + f(a_id_i) + f(a_ia_{i+1}) + f(a_id_{i+1}) + f(w_{3i}) \\
&= i + 4n - i + 3 + 7n - i + 4 + 8n - i + 4 + 10n + i + 3 + 11n + 3i + 3 \\
&= 4n + 7n + 8n + 10n + 11n + i - i - i - i + i + 3i + 3 + 4 + 4 + 3 + 3 \\
&= 40n + 2i + 17
\end{aligned}
$$

Note that the weight of each face is different. Hence, graph DH_n is an (a, d)-face antimagic graph.

Example 3.5 Face antimagic labeling on the double house graph DH_2 with $V(DH_2) = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, d_1, d_2\}, E(DH_2) = \{a_1b_1, a_2b_2, a_3b_3, b_1c_1, b_2c_1, b_2c_2, b_3c_2, a_1d_1, a_2d_1, a_2d_2, a_3d_2, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$, and $\mathcal{F}(DH_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22}), f(w_{31}), f(w_{32})\}$. Then, we define the vertex, edge, and face labeling for the graph using the labeling pattern that has been obtained.

Vertex labeling:
$f(a_1) = 1, f(a_2) = 2, f(a_3) = 3$
$f(b_1) = 6, f(b_2) = 5, f(b_3) = 4$
$f(c_1) = 7, f(c_2) = 8$
$f(d_1) = 10, f(d_2) = 9$
Edge labeling:
$f(a_1b_2) = 11, f(a_2b_2) = 12, f(a_3b_3) = 13$
$f(b_1c_1) = 14, f(b_2c_2) = 15$
$f(a_1d_1) = 17, f(a_2d_2) = 16$
$f(a_1a_{1+1}) = 19, f(a_2a_{2+1}) = 18$
$f(b_1b_{1+1}) = 21, f(b_2b_{2+1}) = 20$

$f(b_{1+1}c_1) = 22, f(b_{2+1}c_2) = 23$
$f(a_1d_{1+1}) = 24, f(a_2d_{2+1}) = 25$
Face labeling:
$f(w_{11}) = 26, f(w_{12}) = 29$
$f(w_{21}) = 27, f(w_{22}) = 30$
$f(w_{31}) = 28, f(w_{32}) = 31$

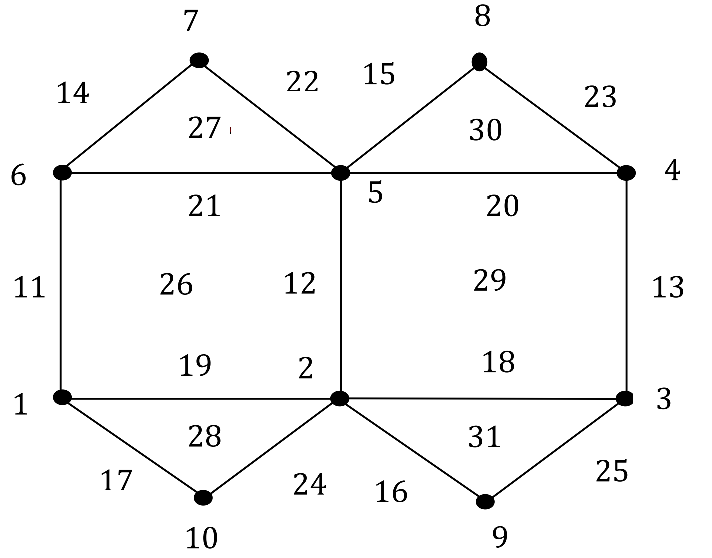

Figure 3.4 Face antimagic labeling on the DH_2 graph

Figure 3.4 illustrates Theorem 3.2 applied to $the\ DH_2$ graph. Given a double house graph with $n = 2$ and $i = 1,2,3$.
Subsequently, the values of $W_{11}, W_{12}, W_{21}, W_{22}, W_{31}, W_{32}$ are obtained as follows.

$W_{11} = 14 + 63 + 26 = 103$
$W_{12} = 14 + 63 + 29 = 106$
$W_{21} = 18 + 57 + 27 = 102$
$W_{22} = 14 + 59 + 30 = 105$
$W_{31} = 13 + 60 + 28 = 101$
$W_{32} = 14 + 59 + 31 = 104$

Based on the results obtained, the set of weights W on the graph resulting from the DH_2 operation is as follows.

$$W = \{101,102,103,104,105,106\}$$

The set W forms an arithmetic sequence with a common difference $d = 1$, so the labeling is called a $\{101, 102, 103, 104, 105, 106\}$-face antimagic labeling.

3.3 Face Antimagic Labeling on Triangular Ladder Graph

As with previous sections, it is important to understand the basic definition of the graph being used for face antimagic labeling. For this last section, the graph is a triangular ladder graph.

Definition 3.7 (Sumathi, 2018) *A triangular ladder graph* (TL_n) *is a graph formed from a ladder graph* (L_n) *by adding an edge connecting vertex* b_i *and vertex* a_{i+1} *for* $1 \leq i \leq n-1$.

Example 3.5. Figure 3.5 illustrates a triangular ladder graph for $n = 2$.

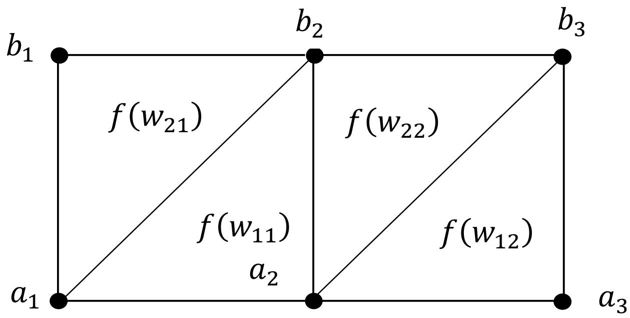

Figure 3.5 Graph TL_2

On this graph, the vertex, edge, and face sets are, in order,
$V(TL_2) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$
$E(TL_2) = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_3, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$
$\mathcal{F}(TL_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22})\}$
Based on the definition of face antimagic labeling on triangular ladder graphs, the following theorem is obtained.

Theorem 3.3 *The triangular ladder graph TL_n is an (a, d)-face antimagic graph.*

Proof. We first construct the labeling functions for the vertices, edges, and faces. Let $g: V(\mathrm{TL_n}) \rightarrow \{1,2, \dots\ n+1\}$ and $dh: n = 1,2,3,4, \dots$. We define the labeling for the vertices, edges, and faces of the triangular ladder graph $\mathrm{TL_n}$ as follows.

Vertex labeling:
$f(a_i) = i$, for $1 \leq i \leq n-1$
$f(b_i) = 2n - i + 3$, for $1 \leq i \leq n-1$
With $i = 1,2, \dots\ n+1$
Edge labeling:
$f(a_ib_i) = 2n + i + 2$, for $1 \leq i \leq n-1$
$f(b_ib_{i+1}) = 3n + i + 3$, for $1 \leq i \leq n-1$
$f(a_ia_{i+1}) = 5n - i + 4$, for $1 \leq i \leq n-1$
$f(a_ib_{i+1}) = 5n + i + 3$, for $1 \leq i \leq n-1$
With $i = 1,2, \dots\ n+1$
Face labeling:
$f(w_{1i}) = 7n + 4 - i$, for $1 \leq i \leq n-1$
$f(w_{2i}) = 8n + 4 - i$, for $1 \leq i \leq n-1$
With $i = 1,2, \dots\ n+1$
Based on the vertex, edge, and face labeling, the face weights are obtained as follows:

$$\begin{aligned} W_{1i} &= f(a_i) + f(b_i) + f(a_ib_i) + f(b_ib_{i+1}) + f(a_ib_{i+1}) + f(w_{1i}) \\ &= i + 2n - i + 3 + 2n + i + 2 + 3n + i + 3 + 5n + i + 3 + 7n + 4 - i \\ &= 2n + 2n + 3n + 5n + 7n + i - i + i + i + i - i + 3 + 2 + 3 + 3 + 4 \\ &= 19n + 2i + 15 \end{aligned}$$

$$\begin{aligned} W_{2i} &= f(a_i) + f(b_i) + f(a_ib_i) + f(a_ia_{i+1}) + f(b_ib_{i+1}) + f(a_ib_{i+1}) + f(w_{2i}) \\ &= i + 2n - i + 3 + 2n + i + 2 + 5n - i + 4 + 3n + i + 3 + 5n + i + 3 + 8n + 4 - i \\ &= 2n + 2n + 5n + 3n + 5n + 8n + i - i + i - i + i + i - i + 3 + 2 + 4 + 3 + 3 + 4 \\ &= 25n + i + 19 \end{aligned}$$

Thus, the weight of each face is different. Hence, graph TL_n admits an (a, d)-face antimagic labeling.

Example 3.6 Face antimagic labeling on the triangular ladder graph TL_2 with $V(TL_2) = \{a_1, a_2, a_3, b_1, b_2, b_3\}, E(TL_2) = \{a_1b_1, a_2b_1, a_2b_2, a_3b_2, a_3b_3, a_1a_2, a_2a_3, b_1b_2, b_2b_3\}$, and $\mathcal{F}(TL_2) = \{f(w_{11}), f(w_{12}), f(w_{21}), f(w_{22})\}$. Then, we define the vertex, edge, and face labeling for the graph using the labeling pattern that has been obtained.
Vertex labeling:
$f(a_1) = 1, f(a_2) = 2, f(a_3) = 3$
$f(b_1) = 6, f(b_2) = 5, f(b_3) = 4$
Edge labeling:
$f(a_1b_1) = 7, f(a_2b_2) = 8, f(a_3b_3) = 9$
$f(b_1b_2) = 10, f(b_2b_3) = 11$
$f(a_1a_2) = 13, f(a_2a_3) = 12$
$f(a_1b_2) = 14, f(a_2b_3) = 15$
Face labeling:
$f(w_{11}) = 17, f(w_{12}) = 16$
$f(w_{21}) = 19, f(w_{22}) = 18$

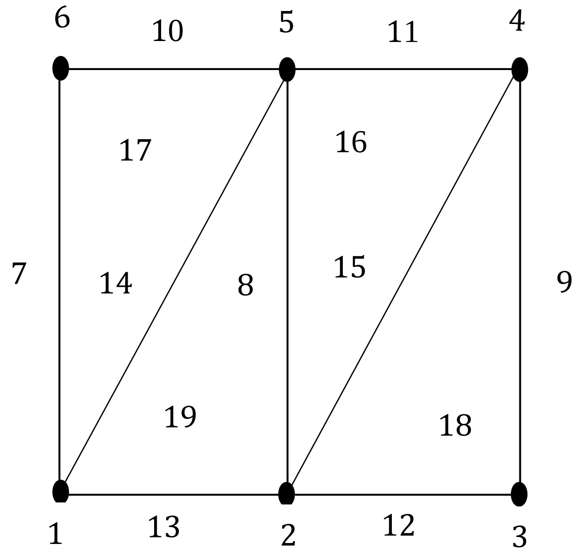

Figure 3.6 Face antimagic labeling on the TL_2 graph
Figure 3.6 illustrates Theorem 3.3. Given a house graph with $n = 2$ and $i = 1,2,3$.
Subsequently, the values of W_{11}, W_{12}, W_{21}, and W_{22} are obtained as follows.
$W_{11} = 12 + 31 = 43 + 17 = 60$
$W_{12} = 11 + 34 = 45 + 16 = 61$
$W_{21} = 8 + 35 = 43 + 19 = 62$
$W_{22} = 9 + 36 = 45 + 18 = 63$

Based on the results obtained, the set of weights W on the graph resulting from the DH_2 operation is as follows.

$$W = \{60,61,62,63\}$$

The set W forms an arithmetic sequence with a common difference $d = 1$, so the labeling is called a $\{60, 61, 62, 63\}$-face antimagic labeling.

CONCLUSION

Based on the results of this research, it can be concluded that house graphs H_n, double house graphs DH_n, and triangular ladder graphs TL_n can be assigned a face antimagic labeling using a specific labeling pattern for their vertices, edges, and faces. This labeling results in the sum of weights on each face forming an arithmetic progression with a common difference of $d = 1$. Furthermore, the labeling construction is bijective and aligns with the definition of face antimagic labeling. Further investigations may consider extending the (a, d)-face antimagic labeling to generalized house graphs or cylindrical ladder graphs.

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