International Journal of Mathematics and Statistics Studies, 12 (4), 1-15, 2024

Print ISSN: 2053-2229 (Print),

Online ISSN: 2053-2210 (Online)

Website: https://www.eajournals.org/

Publication of the European Centre for Research Training and Development -UK

Off Grid Collocation Four Step Initial Value Solver for Second Order Ordinary Differential Equations

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doi: https://doi.org/10.37745/ijmss.13/vol12n4115 Published May 27, 2024

Citation: Alabi, M. O., Raji, M. T. and Olaleye, M. S. (2024) Off Grid Collocation Four Step Initial Value Solver for Second Order Ordinary Differential Equations, *International Journal of Mathematics and Statistics Studies*, 12 (4), 1-15

ABSTRACT: *The derivation and application of a four step Block Linear Multistep Method is hereby presented. To achieve this, Chebyshev polynomial was employed as basis function. Chebyshev polynomial was adopted as basis function based on its level of accuracy among other monomials in the interval [-1, 1]. Block method was adopted in this presentation based on its accuracy over the popular Predictor – Corrector method. The method under consideration gives solution at each grid point within the interval of integration. The method was arrived at by interpolating the polynomial equation and collocating the differential equation at some selected points. The order and error constant of the method were investigated likewise the consistency and zero stability which is one of the desirability property of linear multistep method were equally investigated. The method was applied to solve some second order ordinary differential equations and compare its level of accuracy with the analytical solution and equally compare its level of accuracy with some other existing methods.*

KEYWORDS: collocation, interpolation, block method, predictor, corrector, chebyshev polynomial.

INTRODUCTION

Due to the problems encountered in finding analytical solution to some differential equations, an approximate is sought for. In connection to this, several approximate methods have been proposed over the years, among such include the Runge – Kutta Method, Heuns method, Picard iterative method among others. It is worthy to note that all these methods are single step methods whish was find out that they lack some level of accuracy.

In order to improve upon the level of accuracy of the single step methods, then came the introduction of Linear Multi step Methods. Some of the Multistep methods include the popular Simpson's method which is regarded as the most accurate implicit linear two step multistep method in solving first order ordinary differential equation [16], Also the introduction of predictor – corrector method, that is the Adams Method (Adams Moulton and Adams Bashforth).

As a result of problems encountered in implementing predictor – corrector method, particularly in the level of accuracy of the method, came the introduction of Block methods. The Block method is a self-starting method unlike the predictor – corrector methods which depends on another single step method for its starting value which renders the method to lack some level of accuracy.

This paper presents the derivation and application of Block Linear Multistep method for the direct solution of second order initial value problems of the form

$$y''(x) = f\big(x, y(x), y'(x)\big) \quad (1)$$

together with $y(x_0) = y_0 \ \ and \ \ y'(x_0) = y'_0$

The method is based on interpolating the polynomial equation at some selected grid points while interpolation was done at some selected grid points and an off grid point. Many authors have worked extensively on linear multi - step method to solve equation (1) in which most of them are Predictor – Corrector in nature which is an improvement over and above the known single step methods [1, 2, 3, 4, 5, 7, 9,10, 13]. Howbeit the methods have better accuracy over and above any known single step method but it equally suffers some set back such as it depends on single step methods to determine its starting value apart from being too laborious to develop an appropriate corrector method.

In order to circumvent the hurdle faced in developing appropriate predictor – corrector method, many authors came up with an improved noble method known as Block Linear Multi- step method. This method has an advantage of self-starting which does not depend on additional information in getting its starting value as against the Predictor – Corrector method. Among researchers that have worked extensively on this method include [6, 8, 11,14, 15] to mention but a few.

METHODOLOGY:

In deriving multistep methods as an initial value solver to equation 1, many authors make use of power series method of the form

$$y(x) = \sum_{r=0}^{\infty} a_r x^r \quad (2)$$

where a_r is an arbitrary constant. In this presentation, a linear multistep method of the form

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h^n \sum_{r=0}^{k} \beta_r f_{n+r} + \beta_j f_{n+j} \qquad (3)$$

is hereby proposed where n is the order of the differential equation, both α and β are arbitrary constants and not necessary $\alpha_r = 0$, h is the step length and j is non integer. It is important to note that the smaller the value of h, the better the accuracy of the method [16].

In the course of derivation of the proposed method to solve equation (1), Chebyshev polynomial was used as basis function. The choice of Chebyshev polynomial as basis function over other monomials is its level of accuracy among other monomials in the interval [-1, 1] [12].

DERIVATION OF THE METHOD

Let $\quad y(x) = \sum_{n=1}^{k} a_n T_n(x) \qquad (4)$

Where a_n is an arbitrary constant and $T_n(x)$ is the Chebyshev polynomials which can be generated recessively using the relation

$$T_{n+1} = 2x\, T_n(x) - T_{n-1}(x), \quad n = 0(1)k \qquad (5)$$

where $T_0(x) = 1$ and $T_1(x) = x$ (Fox and Parker).

Here, a four step Block method is hereby proposed. In order to achieve this, the linear multistep of equation (3) was adopted in which interpolation was done at $x = x_{n+r}, r = 0(1)2,$ and collocation was done at $x_{n+r},\ r = 0(1)4$ and $x_{n+\frac{1}{2}},$

To achieve this method, equation (4) together with equation (5) was adopted with $k = 8$ and this leads to the polynomial equation

$$\left.\begin{array}{c} y(x) = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4(8x^4 - 8x^2 + 1) + \\ a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 4x^4 + 18x^2 - 1) + \\ a_7(64x^7 - 112x^5 + 56x^3 - 7x) + a_8(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) \end{array}\right\} \quad (6)$$

Using the shifted Chebyshev polynomial where

$$\left.\begin{array}{c} x = \frac{2x - b - a}{b - a}, x_n \le x \le x_{n+4} \\ hence \\ x = \frac{x - kh - 2h}{2h} \end{array}\right\} \qquad (7)$$

Therefore, equation (6) becomes

$$\left.\begin{array}{c} y(x) = a_0 + a_1\left(\frac{x-kh-2h}{2h}\right) + a_2\left\{2\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} + \\ a_3\left\{4\left(\frac{x-kh-2h}{2h}\right)^3 - 3\left(\frac{x-kh-2h}{2h}\right)\right\} + a_4\left\{8\left(\frac{x-kh-2h}{2h}\right)^4 - 8\left(\frac{x-kh-2h}{2h}\right)^2 + 1\right\} + \\ a_5\left\{16\left(\frac{x-kh-2h}{2h}\right)^4 - 20\left(\frac{x-kh-2h}{2h}\right)^2 + \left(\frac{x-kh-2h}{2h}\right)\right\} + \\ a_6\left\{32\left(\frac{x-kh-2h}{2h}\right)^6 - 48\left(\frac{x-kh-2h}{2h}\right)^4 + 18\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} + \\ a_7\left\{64\left(\frac{x-kh-2h}{2h}\right)^7 - 112\left(\frac{x-kh-2h}{2h}\right)^5 + 56\left(\frac{x-kh-2h}{2h}\right)^3 - 7\left(\frac{x-kh-2h}{2h}\right)\right\} + \\ a_8\left\{128\left(\frac{x-kh-2h}{2h}\right)^8 - 256\left(\frac{x-kh-2h}{2h}\right)^6 + 160\left(\frac{x-kh-2h}{2h}\right)^4 - 32\left(\frac{x-kh-2h}{2h}\right) + 1\right\} \end{array}\right\} \quad (8)$$

The second derivative of equation (8) yields

$$\left.\begin{array}{c} y''(x) = \frac{a_2}{h^2} + \frac{3a_3}{h^3}(x-kh-2h) + a_4\left\{\frac{6}{h^4}(x-kh-2h)^2 - \frac{4}{h^2}\right\} + \\ a_5\left\{\frac{10}{h^5}(x-kh-2h)^3 - \frac{15}{h^3}(x-kh-2h)\right\} + \\ a_6\left\{\frac{15}{h^6}(x-kh-2h)^4 - \frac{36}{h^4}(x-kh-2h)^2 + \frac{9}{h^2}\right\} + \\ a_7\left\{\frac{21}{h^7}(x-kh-2h)^5 - \frac{70}{h^5}(x-kh-2h)^3 + \frac{42}{h^3}(x-kh-2h)\right\} + \\ a_8\left\{\frac{28}{h^8}(x-kh-2h)^6 - \frac{120}{h^6}(x-kh-2h)^4 + \frac{120}{h^4}(x-kh-2h)^2 - \frac{16}{h^2}\right\} + \end{array}\right\} (9)$$

Interpolating equation (8) at $x = x_{n+k}, k = 0(1)2$ and collocating equation (9) at $x = x_{n+k}, k = 0\,(1)4\ and\ at\ x = x_{n+\frac{1}{2}}$ lead to the matrix system of linear equation

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -6 & 20 & -50 & 105 & -196 & 336 \\ 0 & 0 & 32 & -144 & 304 & -360 & 126 & 441 & -1106 \\ 0 & 0 & 1 & -3 & 2 & 5 & -12 & 7 & 12 \\ 0 & 0 & 1 & 0 & -4 & 0 & 9 & 0 & -16 \\ 0 & 0 & 1 & 3 & 2 & -5 & -12 & -7 & 12 \\ 0 & 0 & 1 & 6 & 20 & 50 & 105 & 196 & 336 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ 2y_{n+1} \\ 2y_{n+2} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \\ h^2 f_{n+2} \\ h^2 f_{n+3} \\ h^2 f_{n+4} \end{pmatrix} \quad (10)$$

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Solving the system of equations in equation (10) gives the value of $a_n, n = 0(1)8$

$$a_0 = \frac{1}{23040}\left\{\begin{matrix} 5040y_n - 10080y_{n+1} + 28080y_{n+2} \\ -29h^2f_n - 768h^2f_{n+\frac{1}{2}} + 1676h^2f_{n+1} + 11766h^2f_{n+2} \\ + 5204h^2f_{n+3} + 151h^2f_{n+4} \end{matrix}\right\}$$

$$a_1 = \frac{1}{282240}\left\{\begin{matrix} -35280y_n - 493920y_{n+1} \\ +529200y_{n+2} - 301h^2f_n - 13056h^2f_{n+\frac{1}{2}} - \\ 10612h^2f_{n+1} + 223062h^2f_{n+2} + 114772h^2f_{n+3} \\ +3655h^2f_{n+4} \end{matrix}\right\}$$

$$a_2 = \frac{1}{5760}\left\{\begin{matrix} 5040y_n - 10080y_{n+1} + 5040y_{n+2} - \\ 109h^2f_n - 768h^2f_{n+\frac{1}{2}} - 1844h^2f_{n+1} + 1686h^2f_{n+2} \\ +1684h^2f_{n+3} + 71h^2f_{n+4} \end{matrix}\right\}$$

$$a_3 = \frac{1}{161280}\left\{\begin{matrix} 105840y_n - 211680y_{n+1} + 105840y_{n+2} \\ -3969h^2f_n - 24320h^2f_{n+\frac{1}{2}} - 85764h^2f_{n+1} - 19474h^2f_{n+2} \\ + 25956h^2f_{n+3} + 1731h^2f_{n+4} \end{matrix}\right\}$$

$$a_4 = \frac{1}{5760}\left\{\begin{matrix} 5040y_n - 10080y_{n+1} + 5040y_{n+2} \\ -133h^2f_n - 768h^2f_{n+\frac{1}{2}} - 3188h^2f_{n+1} - 1338h^2f_{n+2} \\ +340h^2f_{n+3} + 47h^2f_{n+4} \end{matrix}\right\}$$

$$a_5 = \frac{1}{161280}\left\{\begin{matrix} 105840y_n - 211680y_{n+1} + 105840y_{n+2} - 3070h^2f_n \\ -24320h^2f_{n+\frac{1}{2}} - 60676h^2f_{n+1} - 19474h^2f_{n+2} + \\ 868h^2f_{n+3} + 835h^2f_{n+4} \end{matrix}\right\}$$

$$a_6 = \frac{1}{40430}\left\{\begin{matrix} 5040y_n - 10080y_{n+1} + 5040y_{n+2} - \\ 77h^2f_n - 768h^2f_{n+\frac{1}{2}} - 3892h^2f_{n+1} - 42h^2f_{n+2} \\ - 364h^2f_{n+3} + 103h^2f_{n+4} \end{matrix}\right\}$$

$$a_7 = \frac{1}{56440}\left\{\begin{matrix} -105840y_n + 211680y_{n+1} - 105840y_{n+2} + \\ 1729h^2f_n + 24320h^2f_{n+\frac{1}{2}} + 63364h^2f_{n+1} + 19474h^2f_{n+2} \\ -3556h^2f_{n+3} + 509h^2f_{n+4} \end{matrix}\right\}$$

$$a_8 = \frac{1}{17920}\left\{\begin{matrix} -1680y_n + 3360y_{n+1} - 1680y_{n+2} + 63h^2f_n \\ +256h^2f_{n+\frac{1}{2}} + 1148h^2f_{n+1} + 238h^2f_{n+2} - 28h^2f_{n+3} + \\ 3h^2f_{n+4} \end{matrix}\right\}$$

Substituting the value of $a_n, N = 0(1)8$ into equation (8) yields the continuous scheme below.

$$y(x) = \frac{1}{1128960}\left\{\begin{matrix} \begin{pmatrix} -13547520x^8 - 13547520x^7 + 31610880x^6 + 35562240x^5 \\ -15805440x^4 - 23708160x^3 - 479647980x \end{pmatrix} y_n + \\ \begin{pmatrix} 27095040x^8 + 27095040x^7 - 63221760x^6 - 71124480x^5 \\ +31610880x^4 + 47416320x^3 - 7902720x \end{pmatrix} y_{n+1} + \\ \begin{pmatrix} -13547520x^8 - 13547520x^7 + 31610880x^6\ 35562240x^5 - \\ 15805440x^4 - 23708160x^3 + 5080320x + 1128960 \end{pmatrix} y_{n+2} + \\ \begin{pmatrix} 508032x^8 + 221312x^7 - 1085056x^6 - 731472x^5 + \\ 529984x^4 + 512736x^3 - 49616x \end{pmatrix} h^2f_n + \\ \begin{pmatrix} 2064384x^8 + 3112960x^7 - 4816896x^6 - 8171520x^5 \\ +2408448x^4 + 5447680x^3 - 733184x \end{pmatrix} h^2f_{n+\frac{1}{2}} + \\ \begin{pmatrix} 9257472x^8 + 8110592x^7 - 22002176x^6 - 20989248x^5 \\ + 11803904x^4 + 13190016x^3 - 1252160x \end{pmatrix} h^2f_{n+1} + \\ \begin{pmatrix} 1919232x^8 + 2492672x^7 - 3876096x^6 - 6543264x^5 \\ +357504x^4 + 4362176x^3 + 2257920x^2 + 346976x \end{pmatrix} h^2f_{n+2} + \\ \begin{pmatrix} -225792x^8 - 455168x^7 + 125440x^6 + 893760x^5 \\ + 740096x^4 + 206976x^3 - 5824x \end{pmatrix} h^2f_{n+3} + \\ \begin{pmatrix} 24192x^8 + 65152x^7 + 43904x^6 - 20496x^5 - 34496x^4 \\ - 11424x^3 + 368x \end{pmatrix} h^2f_{n+4} \end{matrix}\right\}$$

Where $x = \frac{x-kh-2h}{2h}$.

Further simplification of the continuous scheme above leads to the following:

$$\alpha_0(t) = \frac{1}{2}(-24t^8 - 24t^7 + 56t^6 + 63t^5 - 28t^4 - 42t^3 + 5t)$$
$$\alpha_1(t) = (24t^8 + 24t^7 - 56t^6 - 63t^5 + 28t^4 + 42t^3 - 7t)$$

$$\alpha_2(t) = \frac{1}{2}(-24t^8 - 24t^7 + 56t^6 + 63t^5 - 28t^4 - 42t^3 + 9t + 2)$$
$$\beta_0(t) = \frac{1}{10080}(4536t^8 + 1976t^7 - 9688t^6 - 6531t^5 + 4732t^4 + 4578t^3 - 443t)$$
$$\beta_{\frac{1}{2}}(t) = \frac{1}{2205}(4032t^8 + 6080t^7 - 9408t^6 - 15960t^5 + 4704t^4 + 10640t^3 - 1432t)$$
$$\beta_1(t) = \frac{1}{2520}(20664t^8 + 18104t^7 - 49112t^6 - 46851t^5 + 26348t^4 + 29442t^3 - 2795t)$$
$$\beta_2(t) = \frac{1}{5040}(8568t^8 + 11128t^7 - 17304t^6 - 29211t^5 + 1596t^4 + 19474t^3 + 10080t^2 + 1549t)$$
$$\beta_3(t) = \frac{1}{2520}(-504t^8 - 1016t^7 + 280t^6 + 1995t^5 + 1652t^4 + 462t^3 - 13t)$$
$$\beta_4(t) = \frac{1}{70560}(1512t^8 + 4072t^7 + 2744t^6 - 1281t^5 - 2156t^4 - 714t^3 + 23t)$$

Evaluating $x = x_{n+4}$ in $\frac{x-kh-2h}{2h}$, $then$ $t = 1$and this leads to the discrete scheme

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n = \frac{h^2}{420}\left\{25f_{n+4} + 476f_{n+3} + 490f_{n+2} - 700f_{n+1} - 256f_{n+\frac{1}{2}} - 35f_n\right\} \quad (11)$$

First derivatives yields

$$\alpha'_0(t) = \frac{1}{2h}\{-96t^7 - 84t^6 + 168t^5 + 157.5t^4 - 56t^3 - 63t^2 + 2.5\}$$
$$\alpha'_1(t) = \frac{1}{h}\{96t^7 + 84t^6 - 168t^5 - 157.5t^4 + 56t^3 + 63t^2 + 3.5\}$$
$$\alpha'_2(t) = \frac{1}{2h}\{-96t^7 - 84t^6 + 168t^5 + 157.5t^4 - 56t^3 - 63t^2 + 4.5\}$$
$$\beta'_0(t) = \frac{1}{10080h}\{18144t^7 + 6916t^6 - 29064t^5 - 16327.5t^4 + 9464t^3 + 6867t^2 - 221.5\}$$
$$\beta'_{\frac{1}{2}}(t) = \frac{1}{2205h}\{16128t^7 + 21280t^6 - 28224t^5 + 39900t^4 + 9408t^3 + 15960t^2 - 716\}$$
$$\beta'_1(t) = \frac{1}{2520h}\{82656t^7 + 63364t^6 - 147336t^5 - 117127.5t^4 + 52696t^3 + 44163t^2 - 1397.5\}$$
$$\beta'_2(t) = \frac{1}{5040h}\{34272t^7 + 38948t^6 - 51912t^5 - 146055t^4 + 3192t^3 + 29211t^2 + 10080t + 1549\}$$

$$\beta'_3(t) = \frac{1}{2520h}\{-2016t^7 - 3556t^6 + 840t^5 + 4987.5t^4 + 3304t^3 + 693t^2 - 6.5\}$$

$$\beta'_4(t) = \frac{1}{70560h}\{6048t^7 + 14252t^6 + 8232t^5 - 3202.5t^4 - 4312t^3 - 1071t^2 - 11.5\}$$

Evaluating the first derivatives at $x = x_n, x_{n+1}$ *and* x_{n+2} yields the following schemes

$$hy'_n + \frac{3}{2}y_n - 2y_{n+1} + \frac{1}{2}y_{n+2} = \frac{h^2}{70560}\left\{22f_{n+4} - 280f_{n+3} - 101854f_{n+2} + 27608f_{n+1} + 2531584f_{n+\frac{1}{2}} - 9170f_n\right\} \quad (12)$$

$$hy'_{n+1} + \frac{71}{64}y_n - \frac{39}{32}y_{n+1} + \frac{7}{64}y_{n+2} = \frac{h^2}{2257920}\left\{25f_{n+4} - 532f_{n+3} - 2028152f_{n+2} + 613172f_{n+1} + 5816576f_{n+\frac{1}{2}} + 37261f_n\right\} \quad (13)$$

$$hy'_{n+2} - \frac{5}{4}y_n + \frac{7}{2}y_{n+1} - \frac{9}{4}y_{n+2} = \frac{h^2}{141120}\left\{23f_{n+4} - 364 + 21686f_{n+2} - 78260f_{n+1} - 45824f_{n+\frac{1}{2}} - 3104f_n\right\} \quad (14)$$

$$hy'_{n+4} - \frac{29}{2}y_n + 30y_{n+1} - \frac{31}{2}y_{n+2} = \frac{h^2}{70560}\left\{19958f_{n+4} + 118888f_{n+3} + 1130010f_{n+2} - 643496f_{n+1} + 2359552f_{n+\frac{1}{2}} - 29554f_n\right\} \quad (15)$$

Writing equations (11) to (15) in block form defined by

$$Ay_m = hBF(y_m) + ey_n + hdf_n$$

Where $A = (a_{ij}),\ B = (b_{ij}), e = (e_1, \ e_2, \ \ldots \ e_n)^T,\ d = (d_1, \ d_2, \ \ldots \ d_n)^T,$

$$y_n = (y_{n+1}, \ y_{n+2}, \ \ldots \ y_{n+k})^T, F(y_n) = (f_{n+1}, \ f_{n+2}, \ \ldots \ f_{n+k})^T,$$

Thus

$$A = \begin{pmatrix} 8 & -6 & 1 & 0 & 0 & 0 \\ -2 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{-39}{32} & \frac{7}{64} & 0 & 1 & 0 & 0 \\ \frac{7}{2} & \frac{-9}{4} & 0 & 0 & 1 & 0 \\ 30 & \frac{-31}{2} & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{-64}{105} & \frac{5}{3} & \frac{7}{6} & \frac{119}{105} & \frac{5}{84} \\ \frac{79112}{2205} & \frac{493}{1260} & \frac{-145507}{10080} & \frac{-1}{252} & \frac{11}{35280} \\ \frac{22721}{8820} & \frac{153293}{564480} & \frac{-253519}{282240} & \frac{-133}{564480} & \frac{5}{451584} \\ \frac{-716}{2205} & \frac{-3913}{7056} & \frac{1549}{10080} & \frac{-13}{5040} & \frac{23}{141120} \\ \frac{73736}{2205} & \frac{-80437}{8820} & \frac{37667}{2352} & \frac{14861}{8820} & \frac{9979}{35280} \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 0 & 0 \\ \frac{-3}{2} & -1 & 0 \\ \frac{-71}{64} & 0 & 0 \\ \frac{5}{4} & 0 & 0 \\ \frac{29}{2} & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{-1}{12} & \frac{-131}{1008} & \frac{5323}{322560} & \frac{-97}{4410} & \frac{-2111}{5040} \end{pmatrix}^T$$

At this point, some analysis of the method will be carried out such as order, error constant consistent and zero stability of the method. For any linear multistep method to converge.

“For any Linear Multistep Method to converge, it must be consistence and zero stable, in which consistence deals with the order of the multistep method which must be greater than one and controls the magnitude of the local truncation error, while zero stability controls the manner in which the error is propagated as the calculation proceeds”. [16]

At this point, the order and error constant of the method derived shall be investigated. Consider the Linear Multistep Method

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h^n \sum_{r=0}^{k} \beta_r f_{n+r} + \beta_j f_{n+j} \quad (16)$$

associated with the linear difference operator

$$\mathcal{L}[y(x);h] = \sum_{j=0}^{k} \left[a_j y(x+jh) - h^2 \beta_j y''(x+jh)\right]$$

such that $y(x)$ is an arbitrary function, continuous and differentiable in a given interval $[a, b]$ and with the assumption that $y(x)$ has higher derivatives as many as possible. On expanding using Tailor’s series about the point x leads to

$$\mathcal{L}[y(x);h] = C_0 y(x) + C_1 h y^{(1)} x + \ . \ . \ . + C_q h^q y^{(q)}(x) + \ . \ . \ .$$

In which

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_1 = \sum_{j=0}^{k} j\alpha_j, \quad C_2 = \frac{1}{2!} \sum_{j=0}^{k} j^2 \alpha_j - \sum_{j=0}^{k} \beta_j - \beta_v$$

$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j\alpha_j - \frac{1}{(q-2)!}\sum_{j=0}^{k} j^{q-2}\beta_j + v^{q-2}\beta_v \quad ; q = 3(1)k$$

The linear multistep method (16) is said to the of order p, if

$$C_0 = C_1 = C_2 = \ . \ . \ . = C_p = C_{p+1}, but\ C_{p+2} \neq 0$$

Here, C_{p+2} is the error constant. Applying the above to calculate the order and error constant of the derived method, it was discovered that the method if of order 8 with error constant of $C_{p+2} = -\frac{613}{4838}$

Here a theorem shall be stated without proof in regarding the convergence of the Linear Multistep Method.

THEOREM: The necessary and sufficient condition for a Linear Multistep Method to be convergent are that it must be both consistence and zero stable.
Note that consistence controls the magnitude of the local truncation error committed at each stage of the calculation while zero stability controls the manner in which this error is propagated as the calculation progresses [16].

DEFINITION 1 : For a Linear Multistep Method to be consistent, it must have order $p \geq 1$

DEFINITION 2: Any Linear Multistep Method will be zero stable if no root of the first characteristics polynomial $\rho(\zeta)$ has modulus greater that one and every roots with modulus one is simple.
At this juncture, it is concluded that the method derived will be convergent since it satisfies the condition of Theorem 1 stated above.

COMPUTATION EXPERIMENT

Some computation experiment shall be presented in this section so as to demonstrate the level of accuracy of the method.

ILLUSTRATION 1 The study of a vibrating string with damping force can be represented with the differential equation $mx''(t) + bx'(t) + kx(t) = 0$ with $x(0) = x_0,\ x'(0) = v_0$, where m is the mass of the string system, b is the damping constant, k is the spring constant, x_0 is the initial displacement, v_0 is the initial velocity while $x(t)$ is the displacement from equilibrium of the spring system at time t.
Let $m = 36kg, b = 12kg\ sec^{-1}, k = 37kg\ sec^{-2}, x_0 = 70\ cm$ and $v_0 = 10cm\ sec^{-1}$.
Use the following data to find an appropriate expression for the motion of the spring.

ANALYTICAL SOLUTION is $x(t) = 70e^{\frac{-t}{6}}\cos t + 21.67e^{\frac{-t}{6}}\sin t$

TABLE 1: TABLE OF ERROR TO ILLUSTRATION 1

t	y - Exact	y - Computed	Absolute Error
0	70.0000000000	69.9998769487	1.230513 E-4
0.1	70.6267059500	70.6265873486	1.186014 E – 4
0.2	70.5195564800	70.5195768442	2.03642 E – 5
0.3	69.7036923900	69.7036587235	3.36665 E – 5
0.4	69.2105881100	69.2105349621	5.31479 E – 5
0.5	66.0775261500	66.0775106852	1.54648 E – 5
0.6	63.3470301300	63.3470026852	2.74448 E – 5
0.7	60.0662634900	60.0662895134	2.60234 E – 5
0.8	56.2864010700	56.2864009283	1.41700 E – 7
0.9	52.0619806900	52.0619756482	5.04180 E – 6
1.0	47.4502421500	47.4502418975	2.52500 E – 7
1.1	42.5104604700	42.5104826285	2.21585 E – 5
1.2	37.3032805900	37.3032818463	1.25630 E – 6
1.3	31.8900598900	31.8900145984	4.52916 E – 5
1.4	26.3322251700	26.3322981051	7.29351 E – 5
1.5	20.6906499900	20.6906410597	8.93030 E – 6

ILLUSTRATION 2: Consider the differential equation:

$$x^2y'' - 3xy' + 3y = 9(\ln x)^2 + 4,\ \ y(1) = 6, y'(1) = 8,$$

THE ANALYTICAL SOLUTION is $y(x) = 2x^3 - 6x + 3(\ln x)^2 + 8\ln x + 10$

TABLE 2: TABLE OF ERROR TO ILLUSTRATION 2.

y(x)	y - Exact	y - Computed	Absolute Error
0.1	6.8870135870	6.8870019783	1.16087 E – 5
0.2	3.7113678820	3.7113738147	5.93270 E – 6
0.3	2.9708691060	2.9708675381	1.56790 E – 6
0.4	2.9164402610	2.9164768355	3.65745 E – 5
0.5	3.1461815970	3.1461218347	5.97623 E – 5
0.6	3.5282234640	3.5282210589	2.40510 E – 6
0.7	4.0142514950	4.0142509267	5.68300 E – 7
0.8	4.5882307230	4.5882302486	4.74400 E – 7
0.9	5.2484183900	5.2484190283	6.38300 E – 7
1.0	6.0000000000	5.9998742156	1.257844 E – 4
1.1	6.8517335300	6.8517951063	6.15763 E – 5
1.2	7.8142959050	7.8142990217	3.11670 E – 6
1.3	8.8994191380	8.8994902382	7.11002 E – 5

1.4	10.119418590	10.1194103657	8.22430 E – 6
1.5	11.486926730	11.4869230282	3.70180 E – 6
1.6	13.014739270	13.0147302138	9.05620 E – 6
1.7	14.715725030	14.7157268203	1.79030 E – 6
1.8	16.602772810	16.6027715645	1.24550 E – 6
1.9	18.688760320	18.6887721036	1.17836 E – 5
2.0	20.986536490	20.9865302156	6.27063 E – 6

ILLUSTRATION 3: Determine an appropriate expression to represent the solution to the differential equation $3x^2y'' + 11y' - 3y = 8 - 3\ln x$; $y(1) = 1, y'(1) = \frac{4}{3}$.

THE ANALYTIC SOLUTION IS

$$y\,(\,x) = x^{\frac{1}{3}} + \ln x$$

TABLE 3: TABLE OF ERROR TO ILLUSTRATION 3

y(x)	y - Exact	y - Computed	Absolute Error
0.1	-1.8384262100	-1.8384201162	6.0937 E – 6
0.2	-1.0246343650	-1.0246332902	1.0748 E – 6
0.3	-0.5345398542	-0.5345226704	1.71838 E – 5
0.4	-0.1794844321	-0.1794900132	8.43001 E – 5
0.5	0.1005533454	0.1005594732	6.12780 E – 6
0.6	0.3326070415	0.3326005643	6.4772 E – 6
0.7	0.5312290578	0.5312203676	8.6902 E – 6
0.8	0.7051742154	0.7051706673	3.5481 E – 6
0.9	0.8601288689	0.8601277102	1.1587 E – 6
1.0	1.0000000000	0.9998793371	1.206629 E – 4
1.1	1.1275902950	1.1275890455	1.2495 E – 6
1.2	1.2449801260	1.2449911407	1.10147 E – 5
1.3	1.3537571480	1.3537811583	2.40103 E – 5
1.4	1.4551611790	1.4551690571	7.8781 E – 6
1.5	1.5501793510	1.5501705549	8.7961 E – 6
1.6	1.6396107250	1.6396195517	8.8267 E – 6
1.7	1.7241114430	1.7241190844	7.6412 E – 6
1.8	1.8042270640	1.8042299012	2.8372 E – 6
1.9	1.8804162160	1.8804108057	5.4103 E – 6
2.0	1.9530682300	1.9530600984	8.1316 E – 6

ILLUSTRATION 4: Solve completely the differential equation:

$$y'' - \frac{2}{x}\,y' = x^4 - 3x^3 + x^2 :\;\; y(2) = 6\,, y'(1) = 4$$

THE ANALYTICAL SOLUTION IS:

$$y(x) = \frac{x^6}{18} - \frac{3x^5}{10} + \frac{x^4}{4} + \frac{25x^3}{18} - \frac{46}{15}$$

TABLE 4: TABLE OF ERROR TO ILLUSTRATION 4

y(x)	y - Exact	y - Computed	Absolute Error
0.1	-3.0652557222	-3.0652508263	4.8959 E – 6
0.2	-3.0552480000	-3.0552439267	4.0733 E – 6
0.3	-3.0278301670	-3.0278399105	9.7435 E – 6
0.4	-2.9742222222	-2.9742230835	8.6128 E – 7
0.5	-2.8859375000	-2.8859071428	3.03572 E – 5
0.6	-2.7550026667	-2.7550109387	8.272 E – 6
0.7	-2.5741377222	-2.5741309529	6.7693 E – 6
0.8	-2.3368960000	-2.3368099167	8.60833 E – 5
0.9	-2.0377641670	-2.0377601272	4.0398 E – 6
1.0	-1.6722222222	-1.6722230471	8.249 E – 7
1.1	-1.2367635000	-1.2367602715	3.2285 E – 6
1.2	-0.7288746667	-0.7288701526	4.5141 E – 6
1.3	-0.1469757222	-0.1469710928	4.6294 E – 6
1.4	0.5096800000	-0.5096813275	1.3275 E – 6
1.5	1.2411458333	1.2411409034	4.9299 E – 6
1.6	2.0469617778	2.0469618638	8.6 E – 8
1.7	2.9263745000	2.9263709175	3.5825 E – 6
1.8	3.8785973333	3.8785973872	5.39 E – 8
1.9	4.9031102780	4.9031190183	8.7403 E – 6
2.0	6.0000000000	5.9998609201	1.390799 E – 4

DISCUSSION OF RESULTS:

Considering the results presented in the Tables 1 to 4, it is oblivious that the difference between the computed results and the analytical solution is very infinitesimally small. The comparison shows clearly that the approximate solution is not too far from the true solution. This shows that the method derived can be applied in solving directly any second order ordinary differential equations in respective of the type of the differential equation. The method was applied to both homogeneous, inhomogeneous and some special type of second order ordinary differential equations.

From the above inference, it is hereby concluded that the method derived is very adequate in solving any second order ordinary differential equations directly without any need of resolving such some differential equations to a system of first order ordinary differential equations

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