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Modeling Power Exponential Error Innovations with Autoregressive Process

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ABSTRACT: *The regular gussian assumption of the error terms is employed in dynamic time series models when the underlying data are not normally distributed, this often results in incorrect parameter estimations and forecast error. As a result, this paper developed maximum likelihood method of estimation of parameters of an autoregressive model of order 2 [AR (2)] with power-exponential innovations. The performance of the parameters of AR (2) in comparison to normal error innovations was evaluated using the Akaike information criterion (AIC) and forecast performance metrics (RMSE and MAE). Both real data sets and simulated data with different sample sizes were used to validate the models. The results revealed that, it is more appropriate and efficient to model non-normal time series data using AR (2) exponential power error innovations.*

KEYWORDS: maximum likelihood estimation, autoregressive process, innovations, power exponential.

INTRODUCTION

Autoregressive time series modeling is a popular statistical technique used to analyze and forecast time series data. Literarily, time series analysis is an important field of statistics and econometrics that deals with the modeling and prediction of temporal data. Autoregressive (AR) models are a class of time series models that use the past values of a series to predict its future values. AR models have been extensively studied and applied in various fields, including finance, engineering, and climate science (Box et al., 2015; Lütkepohl, 2005).

One important aspect of such modeling is the specification of the error term, which describes the discrepancy between the predicted values and the actual values of the time series. However, the standard AR models assume that the errors or residuals of the model are normally distributed with constant variance. This assumption may not hold in many real-world situations, where the errors may exhibit non-constant variance or non-normality. To address these issues, various generalized AR models have been proposed, including the autoregressive conditional heteroscedasticity (ARCH)

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model (Engle, 1982) and its extensions, such as the generalized autoregressive conditional heteroscedasticity (GARCH) model (Bollerslev, 1986), It is belief that standard autoregressive models often assume that the error terms are normally distributed, which may not always be the case in real-world applications. To address this limitation, a new class of autoregressive models, called power exponential autoregressive models is developed.

LITERATURE/THEORETICAL UNDERPINNING

In recent years, there has been growing interest in using power exponential error innovations in autoregressive time series models, as these have been shown to provide a more flexible and accurate way of modeling the error term. For instance, a study that explores the use of power exponential error innovations in autoregressive time series modeling is "Autoregressive time series modeling with power exponential error innovations" by Doherty et al. (2019). In this study, the authors develop a new class of autoregressive time series models that incorporate power exponential error innovations. They demonstrate the effectiveness of these models in capturing the complex dynamics of real-world time series data, such as financial data and environmental data. The study also provides a comprehensive theoretical analysis of the properties of the new models, including the asymptotic behaviour of the estimators and the optimal choice of the model parameters. The authors also compare the performance of the new models with that of existing models, such as the autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models.

Another study by Liu *et al*. (2021) aimed to extend the power exponential autoregressive model to include error innovations that follow a power exponential distribution. The study proposed a new model, called the Power Exponential Error Power Exponential Autoregressive (PEEPEAR) model, and evaluated its performance on simulated and real-world data. The authors found that the PEEPEAR model outperformed other commonly used models in terms of both forecasting accuracy and robustness to non-normal error distributions. The study has important implications for improving the accuracy of time series forecasting in a variety of fields, including finance, economics, and engineering.

Okamura et.al proposed a New robust approach that estimates autocorrelation accurately and reduced the influence of outliers. He then compares his results with the conventional least square and least absolute deviation method. His results show that the new method provides unbiased autocorrelation for highly contaminated simulated data with extreme outliers over other methods. However, the present study aims to use Maximum Likelihood Method in estimating parameters of AR models in the presence of normality assumption violation. We extend the standard AR model by allowing for the Power Exponential distribution of the errors, which provides more flexibility in modelling the tails of the distribution and allows for skewness and kurtosis. We also compare the performance of the model with the standard AR model using real-world financial data.

METHODOLOGY

This paper makes use of maximum likelihood method of parameters estimation as folllows:

Estimation of Parameters

Estimation of parameters of AR (2) with Exponential power error innovations:

This is also known as generalized normal distribution. It allows β and σ to be any positive real numbers and μ to be any real number.

If G is a random variable from a power exponential distribution, its probability density function is given by the following

$$f(g,\mu,\sigma,\beta) = \frac{1}{\sigma\Gamma\left(1+\frac{1}{2\beta}\right)2^{\left(1+\frac{1}{2\beta}\right)}} \exp\{-\frac{1}{2}\left|\frac{g-\mu}{\sigma}\right|^{2\beta}\} \quad 1$$

$$-\infty < \mu < \infty \ and \ \sigma > 0$$

Where σ^2 scale parameter, β is the shape parameter and μ is the location parameter

If X_t follows autoregressive model of order two AR (2), we have

$X_t = \phi_1 X_{t-1} + \phi_1 X_{t-2} + e_t,$

$e_t = X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}$

When error is no longer white noise, using AR (2) with power exponential error innovations, we have

$$f(e_t) = \frac{1}{\sigma\Gamma\left(1+\frac{1}{2\beta}\right)2^{\left(1+\frac{1}{2\beta}\right)}} \exp\{-\frac{1}{2}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_1 X_{t-2}}{\sigma}\right|^{2\beta}\} \quad 2$$

Taking the likelihood of equation 2, we have

$$\prod_{i=1}^{n} f(e_t) = \frac{1}{\sigma^n\Gamma\left(1+\frac{1}{2\beta}\right)^n 2^{n\left(1+\frac{1}{2\beta}\right)}} \exp\sum_{t=3}^{n}\{-\frac{1}{2}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_1 X_{t-2}}{\sigma}\right|^{2\beta}\}$$

The log likelihood is as follows

$$\log\prod_{i=1}^{n} f(e_t) = \log \sigma^{-n}\Gamma\left(1+\frac{1}{2\beta}\right)^{-n} 2^{-n\left(1+\frac{1}{2\beta}\right)+\sum_{t=3}^{n}\{-\frac{1}{2}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_1 X_{t-2}}{\sigma}\right|^{2\beta}\}}$$

$$= -n\log\sigma - n\log\Gamma\left(1+\frac{1}{2\beta}\right) - n\left(1+\frac{1}{2\beta}\right)\log 2 - \frac{1}{2}\sum_{t=3}^{n}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_1 X_{t-2}}{\sigma}\right|^{2\beta} \quad 3$$

Differentiate equation 3 with respect to ϕ_1, ϕ_2, σ and β we have

$$\frac{\partial \log\prod_{i=1}^{n} f(e_t)}{\partial\phi_1} = \frac{\beta}{\sigma}\sum_{t=3}^{n} X_{t-1}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma}\right|^{2\beta-1} = 0 \quad 4$$

$$\frac{\partial \log\prod_{i=1}^{n} f(e_t)}{\partial\phi_2} = = \frac{\beta}{\sigma}\sum_{t=3}^{n} X_{t-2}\left|\frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma}\right|^{2\beta-1} = 0 \quad 5$$

Equation 4 and 5 has no close form but if β=1 the solution can be obtained.

From equation 4

$$\frac{\partial \log\prod_{i=1}^{n} f(e_t)}{\partial\sigma} = = -\frac{n}{\sigma} + \frac{\beta}{\sigma^{2\beta+1}}\sum_{t=3}^{n}\left|X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}\right|^{2\beta-1} = 0 \quad 6$$

Multiply both sides by σ

$$\frac{n\sigma}{\sigma} = \frac{\sigma\beta}{\sigma^{2\beta+1}}\sum_{t=3}^{n}\left|X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}\right|^{2\beta-1}$$

$$\sigma^{2\beta} = \beta\frac{\sum_{t=3}^{n}\left|X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}\right|^{2\beta}}{n} \quad 7$$

When $\beta = 1$, equation 7 becomes

$$\sigma^2 = \frac{\sum_{t=3}^{n} |X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}|^2}{n}$$

From equation 3

$$\frac{\partial \log \prod_{i=1}^{n} f(e_t)}{\partial \beta} = \frac{-n\, \Gamma'\left(1+\frac{1}{2\beta}\right)}{\Gamma\left(1+\frac{1}{2\beta}\right)} + \frac{n \log 2}{2\beta^2} - \sum_{t=3}^{n} \frac{\partial}{\partial \beta} \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|^{2\beta} \qquad 8$$

From 3.3.8, let us consider

$$\sum_{t=3}^{n} \frac{\partial}{\partial \beta} \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|^{2\beta}$$

Let $p = \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|^{2\beta}$ 9

If log is introduced to both sides of 9 we have

$\text{Log } p = 2\beta log \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|$

$\frac{\partial p}{\partial \beta} \text{ x} \frac{1}{p} = 2\beta log \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|$

$\frac{\partial p}{\partial \beta} = 2\beta log \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right| \text{ x } p$

$\frac{\partial p}{\partial \beta} = 2\beta \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|^{2\beta} \text{ x } log \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|$ 10

By putting 10 into 8 we have: $\frac{\partial \log \prod_{i=1}^{n} F(e_t)}{\partial \beta} =$

$$\frac{-n\, \Gamma'\left(1+\frac{1}{2\beta}\right)}{\Gamma\left(1+\frac{1}{2\beta}\right)} + \frac{n \log 2}{2\beta^2} - 2\beta \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|^{2\beta} \text{x} log \left| \frac{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}}{\sigma} \right|$$

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Equation 3, 4 and 11 are solved iteratively using numerical method to obtain maximum likelihood estimates of β, ϕ_1 *and* $\phi_2$because there is no close form solution for the parameters.
Equation 7 has been solved analytically to obtain σ^2.

When β=1, it becomes $\sigma^2 = \frac{\sum_{t=3}^{n} |X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}|^2}{n}$

i.e the variance of AR (2) with normal error term.

RESULTS AND DISCUSSIONS

The summary statistics of the 180 data points were calculated and plotted in charts and diagrams as a form of data cleaning exercise given in table1a. below.

Table 1a.:

MIN	1stQU	MEDIAN	MEAN	3rd QU	MAX
96.96	107.80	118.50	128.00	142.60	216.60

Descriptive Statistics

The data used in validating these models is monthly import commodity price index obtained from Central Bank of Nigeria Statistical Bulletin, 2000-2014 . AIC/BIC criterion was used to determine the suitable order for the model as seen in table 1b below.

Table 1b.

AR	AIC	BIC
1	985.5	991.9
2	947.6	957.2
3	949.6	962.4
4	950.4	966.4
5	952.1	971.2
6	952.7	975.0
7	952.9	978.5

Order Determination Criterion Source: R statistics software

Table 2: ,

Name of the test:	**Shapiro-Wilk Normality Test**
Data:	180
Test statistic:	0.84935
P-value:	2.365e-12

Shapiro-Wilk Normality Test - with p-value 2.365e-12

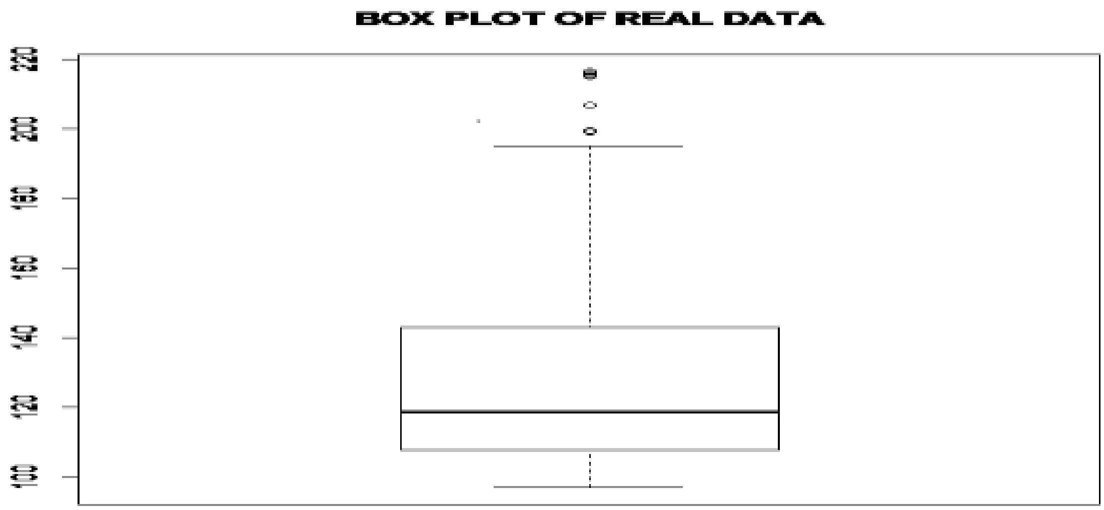

Fig 1. Box plot of price index of import commodity in Nigeria between 2000 and 2014

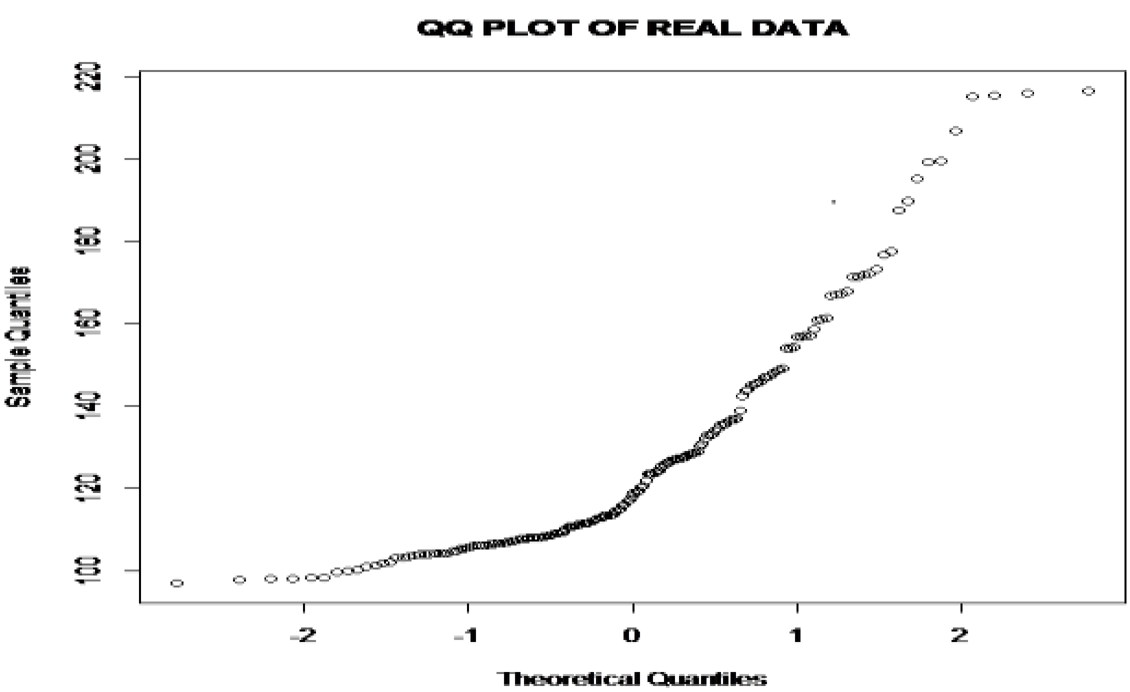

Fig 2. QQ plot of price index of import commodity in Nigeria between 2000 and 2014

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Fig 1 and Fig 2 show that some values stand out in the data set which indicates that there are outliers in the data set.

Table 3:

Coefficient	Estimate	Standard error	t-value	Pr(>t)
$\emptyset_1$	0.4662	0.0666	7.0004	2.552e-12 ***
$\emptyset_2$	0.4606	0.0671	6.7175	1.849e-11 ***

Estimation of parameter of AR (2) with normal error innovationslog likelihood = -726.19 : AIC = 1460.38 Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Source: R Statistics software

Table 4:

Coefficient	Estimate	Standard error	t-value	Pr(>t)
$\emptyset_1$	0.4994	0.2288	2. 182	0.0291
$\emptyset_2$	0.5006	0.2288	2.188	0.0289

Estimation of parameter of AR(2) with power exponential error innovations

log likelihood = -145.8869: AIC = 301.7738

signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Source: R Statistics software

Tables 3 to 4 presented the estimate of parameters of Auto-Regressive model of order two and AIC of each model. It was shown that power exponential error innovation with AIC = 1446.328 is smaller relative to normal error innovation with AIC = 1460.38. It could be deduced that power exponential distribution is superior to normal distributions in terms of dynamic model fitting.

Table 5:

Distribution	AIC	Log lik
Normal	1460.38	-726.19
Power exponential	1446.328	-719.1642

SUMMARY OF RESULTS

From Table 5, it was observed that power exponential error innovations performed better than normal error innovation with non-normal data judging from their AIC.

Forecast Performance

Table 6:

INNOVATIONS	RMSE	MAE
NORMAL	7.6795	5.1828
POWER EXPONENTIAL	7.0418	4.7574

Error Measures

From table 6, the error measures of each indicated that power exponential error innovations is an efficient model for forecasting with the least RMSE and MAE than normal error innovation judging from the error measures.

RMSE: - Root Mean Square Error; **MAE:** - Mean Absolute Error;

Implication to Research and Practice

The implication of this research is that it will assist researchers achieve great precision and reduce frcast error hence quality of life is enhanced. It will also settle methodological dispute among the users of distributions. It also offers chance to verify and improve theories, spot knowledge gaps or limitations, and develop fresh research question.

CONCLUSION

The paper examines the Autoregressive model AR (2) with Power-Exponential error innovations. It uses Maximum Likelihood Estimation to derive parameters and compares its performance with normal error innovations using AIC and forecast performance criteria. Results indicate that AR (2) models with exponential power error innovations are more suitable and efficient for modeling non-normal time series processes.

Future Research

Autoregressive time series of higher order will be considering in the subsequent research.

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