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Deriving an equation of electric potential of a point P separated from a charged flat plate

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ABSTRACT: *An equation of electric potential of a particle P separated from a uniformly charged flat plate was theoretically derived. An equation of electric field was also introduced from this equation. The electric potential is expressed by equation comprised of the logarithmic function and tangential inverse trigonometric function, which include elements (i.e. oblique line and sides of a base) of the quadrangular pyramid PABCO created by connecting the point P with respective vertexes of a rectangle and irrational function. The equation of the electric potential produced when the point P is just above the center of the flat plate is expressed as equation four times the equation symmetric with respect to a and b which are lengths of the sides in the base of the pyramid*

KEYWORDS: electric potential by flat plate, electric field by flat plate, theoretical analysis of complex double integration

INSTRUCTIONS

A calculation example of electric potential due to electric charge distribution which is basis of electromagnetism is introduced into a textbook of a primary physics. Calculation examples of electrification of a particle, such as electric potential due to three-point charges, electric potential due to an electric dipole, electric potential due to an electric quadrupole and electric potential due to a uniformly charged circular disk are described in "Physics for students of science and engineering" by Halliday and Resnick [1]. Other calculation examples are also required in addition to these calculation examples. For example, there are many applications using a plate of which an electrode is not circular but flat, that is to say, electrodes of high frequency dielectric heating used to adhesion of timbers or plastics materials, etc., electrodes of a measuring

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device for electrorheological fluids in shear mode, electrodes of the manufacturing of sandpaper, and vertical and horizontal pole plates which control deviation of an electron-gun in Braun tube used to a TV and an oscilloscope and so on. Theoretical analysis is absolutely necessary for the applications to be designed to have high performance. The example of approximation theoretical analysis for a charged flat plate is explained in fundamental lecture described in Journal of the Institute of Electrostatics Japan as a rarely analytical example [2]. Since the equation of the theoretical analysis cannot be integrated simply, the solution of the equation is executed by using approximation method. In addition, solution of strength of electric field concerning an infinitely wide flat plate is presented in the textbook by using the law of Coulomb [3]. However, the calculation examples of the electric potential and electric field which have rigorously analytical solution due to the charged flat plate are not found in the literatures which have already been published. Therefore, it is the present situation to have been conducting practical use, not by theoretical analysis, but by experimental means or by numerical analysis [4]. Consequently, the equation of the electric potential due to the uniformly charged flat plate is considered and then the equation of electric field is introduced therefrom and both of the solutions can be presented. Thus, it is aim to realize theoretical contribution to the applications.

Theory

Electric potential due to a uniformly charged flat plate

On the occasion of analysis about electric potential in all points within space, explanation is advanced by picking up an arbitrary point on the flat plate as easy analytical method. The electric potential in point P with distance z vertically separated from origin O of the center axis of the flat plate (2l×2w) is determined when the charge is uniformly distributed on the flat plate with area charge density σ(vertical axis passing through the origin O is defined as Z-axis). As shown in Figure 1, an inclined segment is drawn from point P, which extends right above from the origin O of the flat plate and is separated by distance z, on infinitesimal area dx·dy, which is formed on x-y plane and is separated by distance y from Y-axis and by distance x from X-axis, its distance is defined as R. The distance R is expressed as follows:

$$R = \sqrt{x^2 + y^2 + z^2} \qquad (1)$$

The electric potential V_G that the infinitesimal area dx·dy applies on the point P is given by

$$V_G = \frac{\sigma}{4\pi\varepsilon_0} \iint \frac{dx \cdot dy}{R} \qquad (2)$$

Substituting equation (1) into equation (2), we have

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$$V_G = \frac{\sigma}{4\pi\varepsilon_0}\int_{-w}^{w} dy \int_{-l}^{l} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dx \tag{3}$$

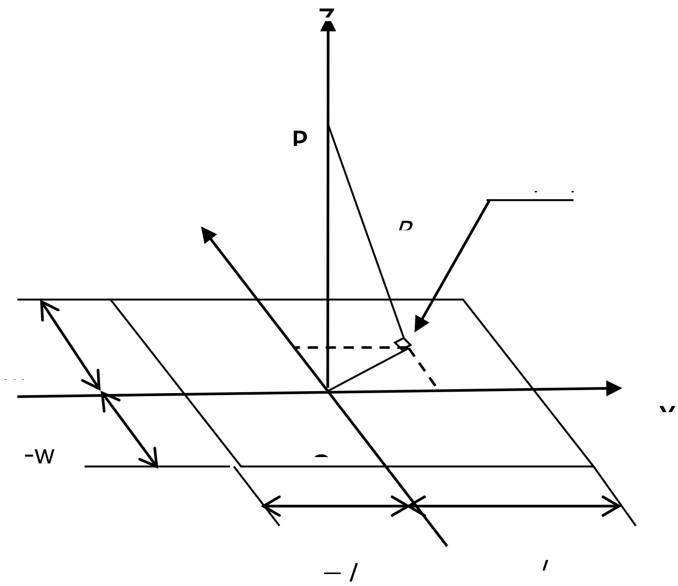

Figure 1 Electric potential of point P due to uniformly charged flat plate

As the equation (3) is integral of the infinitesimal area dx·dy in four quadrants, a typical solution of double integral at the first quadrant in x-y plane is determined in order to have versatility about the equation and make its calculation easier and then the solution is applied to other quadrants. Thus, the integral of subdivision into four in the equation (3) is solved by referring to Figure 2. We have

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_{0}^{b} dy \int_{0}^{a} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dx \tag{4}$$

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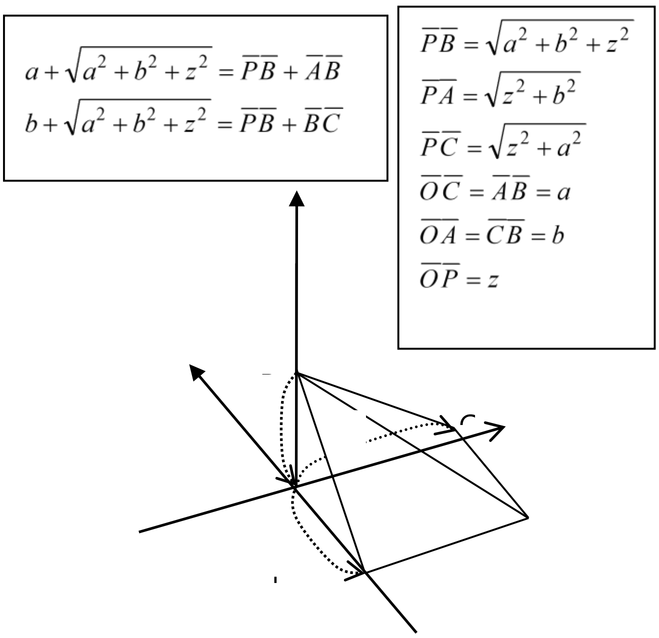
First, the integral with respect to x is solved. The indefinite integral of $\int \frac{1}{\sqrt{x^2+y^2+z^2}}dx$ is solved using integration by substitution. The substitution variable t is defined as $t=\sqrt{x^2+y^2+z^2}+x$. The equation (4) is given by

$$\int \frac{1}{t}dt = \log t$$



The definite integral of above equation is expressed as

$$\int_0^a \frac{1}{\sqrt{x^2+y^2+\dots}} \qquad (5)$$

Figure 2. Comparison of numerical formula in equation (15) with respective edges of quadrilateral pyramid

Next, the equation (5) is integrated with respect to y. We begin with solution concerning the second term of the equation (5) using partial integration and have

$$\int \log\left(\sqrt{y^2+z^2}\right)dy = y\log\left(\sqrt{y^2+z^2}\right) - y + z\tan^{-1}\frac{y}{z} \qquad (6)$$

Furthermore, the indefinite integral of the first term of the equation (5) is solved using partial integration and integration by substitution and is given by

$$\int \log\left(a+\sqrt{y^2+z^2+a^2}\right)dy = y\log\left(a+\sqrt{y^2+z^2+a^2}\right) - \int \frac{y^2}{\left(a+\sqrt{y^2+z^2+a^2}\right)\sqrt{y^2+z^2+a^2}}dy \tag{7}$$

The second term of the equation (7) is solved and we obtain

$$\int \frac{y^2}{\left(a+\sqrt{y^2+z^2+a^2}\right)\sqrt{y^2+z^2+a^2}}dy = \int \frac{y^2}{y^2+z^2} \times \frac{\sqrt{y^2+z^2+a^2}-a}{\sqrt{y^2+z^2+a^2}}dy$$

$$= y - z\tan^{-1}\frac{y}{z} - a\log\left(y+\sqrt{y^2+z^2+a^2}\right) + \int \frac{az^2\left(z^2+a^2\right)}{z^2\left(y^2+z^2+a^2\right)+a^2y^2} \times \frac{1}{\sqrt{y^2+z^2+a^2}}dy \tag{8}$$

In order to solve the term of indefinite integral of the equation (8), putting the function of $\frac{ay}{z\sqrt{y^2+z^2+a^2}}$ as variable t, we obtain relationships shown below

$$t^2 = \frac{a^2y^2}{z^2\left(y^2+z^2+a^2\right)} \qquad t^2+1 = \frac{a^2y^2+z^2\left(y^2+z^2+a^2\right)}{z^2\left(y^2+z^2+a^2\right)}, \quad \frac{dy}{dt} = \frac{z}{a} \times \frac{\left(\sqrt{y^2+z^2+a^2}\right)}{z^2+a^2}$$

Substituting these equations into the term of indefinite integral of the equation (8), we have

$$z\int \frac{1}{t^2+1}dt = z\tan^{-1} t$$

Restoring the function t to the function y, we have

$$\int \frac{az^2\left(z^2+a^2\right)}{z^2\left(y^2+z^2+a^2\right)+a^2y^2} \times \frac{1}{\sqrt{y^2+z^2+a^2}}dy = z\tan^{-1}\frac{ay}{z\sqrt{y^2+z^2+a^2}} \tag{9}$$

Consequently, the first term of the right-hand side of the equation (5) is expressed as

$$\int \log\left(a+\sqrt{y^2+z^2+a^2}\right)dy = y\log\left(\sqrt{y^2+z^2+a^2}+a\right) - y + z\tan^{-1}\frac{y}{z} + a\log\left(y+\sqrt{y^2+z^2+a^2}\right) - z\tan^{-1}\frac{ay}{z\sqrt{y^2+z^2+a^2}} \tag{10}$$

Therefore, the equation (5) is given by

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$$f(a,b,z)=\int_0^b\left\{\log\left(\sqrt{a^2+y^2+z^2}+a\right)-\log\left(\sqrt{y^2+z^2}\right)\right\}dy$$

$$=-z\tan^{-1}\frac{ab}{z\sqrt{a^2+b^2+z^2}}-\frac{1}{2}a\log\left(a^2+z^2\right)-\frac{1}{2}b\log\left(b^2+z^2\right)$$
$$+b\log\left(a+\sqrt{a^2+b^2+z^2}\right)+a\log\left(b+\sqrt{a^2+b^2+z^2}\right) \quad (11)$$

The equation (11) forms the equation symmetric with respect to *a* and *b*.

Now, we describe this definite integral as f(a,b,c). The electric potential in the coordinate (x',y',z') for the uniformly charged flat plate which disposes its center as the origin as shown in Figure 1 are solved by applying the symmetric equation (11) about a and b, respectively when the position of the point P in Figure 1 is on the flat plate and out of the flat plate.

$$\frac{4\pi\varepsilon_0}{\sigma}V(x',y',z)=f(l-x',w-y',z)+f(x'+l,w-y',z)$$
$$+f(l-x',y'+w,z)+f(x'+l,y'+w,z)$$

$$(-l<x'<l,-w<y'<w) \quad (12)$$

$$\frac{4\pi\varepsilon_0}{\sigma}V(x',y',z)=f(l+x',w-y',z)+f(x'+l,w+y',z)$$
$$-f(x'-l,w-y',z)-f(x'-l,y'+w,z)$$

$$(-l<x',0<y'<w) \quad (13)$$

$$\frac{4\pi\varepsilon_0}{\sigma}V(x',y',z)=f(l+x',w+y',z)-f(x'-l,w+y',z)$$
$$-f(x'+l,y'-w,z)-f(x'-l,y'-w,z)$$

$$(x'<l,w<y') \quad (14)$$

The above three equations (12), (13), (14) are not presented on the paper because the respective equations become too redundant. Now, the analyzed electric potential to the point P due to the uniformly charged flat plate becomes complex in comparison with analyses of the electric potential to the point P due to the uniformly charged circular disk and the electric potential to the point P due to the uniformly charged segment [5]. Deriving the equation of the electric potential results in whether or not the solution of the double integral about an irrational function is possible. It cannot be addressed simply and cannot be solved if having to make good use of techniques of integration by substitution and partial integration. The equation of the electric potential produced when the point P is just above the center of the flat plate is established when the values of x',y' in the equation (12) are set to zero and corresponds to four times the equation (11). That's to say, we have

$$V(0,0,z)=\frac{\sigma}{\pi\varepsilon_0}\left\{\begin{array}{l}-z\tan^{-1}\dfrac{lw}{z\sqrt{l^2+w^2+z^2}}-\dfrac{1}{2}l\log\left(l^2+z^2\right)-\dfrac{1}{2}w\log\left(w^2+z^2\right)\\+w\log\left(l+\sqrt{l^2+w^2+z^2}\right)+l\log\left(w+\sqrt{l^2+w^2+z^2}\right)\end{array}\right\} \tag{15}$$

Electric field due to the uniformly charged flat plate

Next, we try to solve an electric field from the electric potential. The relationship between the electric potential and the electric field is expressed as $E_z=-\dfrac{dV}{dz}$.

Hence, equation of the electric field E_z generating when the point P is just above center of the flat plate is given by

$$\begin{aligned}\frac{\pi\varepsilon_0}{\sigma}E_z&=\tan^{-1}\left[\frac{lw}{z\sqrt{l^2+w^2+z^2}}\right]-\frac{lw\times\left(l^2+w^2+2z^2\right)}{\left(w^2+z^2\right)\left(z^2+l^2\right)\sqrt{l^2+w^2+z^2}}\\&+\frac{lz}{z^2+l^2}+\frac{wz}{z^2+w^2}-\frac{wz}{\left(l+\sqrt{l^2+w^2+z^2}\right)\sqrt{l^2+w^2+z^2}}\\&-\frac{lz}{\left(w+\sqrt{l^2+w^2+z^2}\right)\sqrt{l^2+w^2+z^2}}\end{aligned} \tag{16}$$

RESULTS AND DISCUSSION

Instinctive meaning of Equation of Electric Potential

It seems that the equation (15) is composed of complex elements but it is easy to understand it instinctively if creating a quadrangular pyramid by connecting the point P with respective points (A,B,C,O) of four corners of the flat plate. The sides of the quadrangular pyramid correspond to elements of the equation (15) (Refer to Figure 2). For example, to approximate the value z to zero (z→0) is to shift the point P along Z-axis in downward direction and thus the quadrangular pyramid is squashed and then becomes the shape of the flat plate. On the contrary, to approach infinity (z→∞) is to shift the point P along Z-axis in upward direction and thus the quadrangular pyramid approaches the shape of a substantially rectangular parallelepiped.

Special cases about the electric potential and the electric field are described below.

Value of Electric Potential at z→0

If we consider the equation (15) mathematically, the value of the electric potential at z→0 is given by

$$\lim_{z\to 0} V(0,0,z) = \frac{\sigma}{\pi\varepsilon_0}\log\left\{ \begin{array}{l} \left(\frac{l}{w}+\sqrt{\left(\frac{l}{w}\right)^2+1}\right)^w \\ \times\left(\frac{w}{l}+\sqrt{\left(\frac{w}{l}\right)^2+1}\right)^l \end{array}\right\} \qquad (17)$$

Value of Electric Potential at z≫1 and z≫w

Next, when the value of z becomes large sufficiently, that is, z≫l and z≫w, we can neglect $\left(\frac{l}{z}\right)^2, \left(\frac{w}{z}\right)^2, \frac{l\cdot w}{z^2}$ and the equation (15) is transformed and its equation can be approximated as

$$\frac{\pi\varepsilon_0}{\sigma}\cdot V(0,0,z) = w\log z\left(\frac{l}{z}+\sqrt{\frac{l^2}{z^2}+\frac{w^2}{z^2}+1}\right)+l\log z\left(\frac{w}{z}+\sqrt{\frac{l^2}{z^2}+\frac{w^2}{z^2}+1}\right)$$
$$-\frac{w}{z}\log z^2\left(\frac{w^2}{z^2}+1\right)-\frac{l}{z}\log z^2\left(\frac{l^2}{z^2}+1\right)-z\tan^{-1}\frac{lw}{z^2\sqrt{\frac{l^2}{z^2}+\frac{w^2}{z^2}+1}}$$

$$\cong w\log\left(\frac{l}{z}+1\right)+l\log\left(\frac{w}{z}+1\right)$$

or (18)

$$w\log(l+z)+l\log(w+z)-(w+l)\log z$$

Value of Electric Potential at z→∞

Furthermore, it is self-evident that the value of V(0,0,z) becomes zero when the value of z approaches infinity.

Value of Electric Field at z→0

Now, we consider special case of the electric field. When the value of z in the equation (16) approaches zero (z→0), the value of the E_z is given by

$$\lim_{z \to 0} E_z = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{\pi\varepsilon_0}\sqrt{\frac{l^2}{w^2} + \frac{w^2}{l^2}} \tag{19}$$

In the case of the electric field E_z for a point P located on the axis of center of a uniformly charged circular disk, the value of $\lim_{z \to 0} E_z$ becomes $\frac{\sigma}{2\varepsilon_0}$ independent of size of shape, such as radius of the circular disk. On the contrary, the value of $\lim_{z \to 0} E_z$ for a point P located above the z-axis of a uniformly charged flat plate is expressed by the equation (19) and is dependent of size of shape, such as width and length of the flat plate.

CONCLUSION

The equation of electric potential of a point P separated from the uniformly charged flat plate was theoretically derived and the equation of electric field was introduced from the equation of the electric potential as well. The equation of the electric potential is expressed by the symmetrical equation with respect to the width *w* and length *l* and is expressed by the equation composed of the logarithmic and tangential inverse trigonometric function, which include elements of the quadrilateral pyramid PABCO (i.e. oblique edges and sides of the base), and irrational function. The closer values x',y' approach zero, the closer they approach the equation (15) of electric potential of result produced when the point P is located above the center of the flat plate.

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