International Journal of Mathematics and Statistics Studies, 13 (1), 36-49, 2025
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development -UK

Total Product Cordial Labeling of Generalized Dragonfly Graph $Dg_n^{(m,k)}$

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doi: https://doi.org/10.37745/ijmss.13/vol13n13649 Published April 27, 2025

Citation: Khoiriyah M., Ratnasari L., and Utomo R.H.S. (2025) Total Product Cordial Labeling of Generalized Dragonfly Graph, *International Journal of Mathematics and Statistics Studies*, 13 (1), 36-49

Abstract. Suppose G is a graph consisting of two finite sets, namely the set of vertex $V(G)$ and the set of edges $E(G)$, denoted by $G = \big(V(G), E(G)\big)$. A graph G is said to be a total product cordial graph if there exists a vertex labeling $f: V(G) \to \{0,1\}$ such that it induces an edge labeling $f^*: E(G) \to \{0,1\}$ defined by $f^*(uv) = f(u)f(v)$ and satisfies $\left|(v_f(0) + e_{f^*}(0)) - (v_f)\right| \leq 1$. In this paper, it will be proved that the generalized dragonfly graph $(Dg_n^{(m,k)})$ is a total product cordial graph.

Keywords: generalized dragonfly graph, total product cordial labeling

INTRODUCTION

In 1736, Leonard Euler introduced graph theory as a puzzle solver for the Konigsberg bridge problem by visualizing the land as the vertex and the bridge as the edge (Powell and Hopkins, 2015). One of the topics discussed in graph theory is graph labeling. Graph labeling is a mapping or function that pairs each element of a graph (vertex, edge, or both) to an integer under certain conditions. One example of graph labeling that has been widely developed is cordial labeling. Cordial labeling was first introduced in 1987 by Cahit, in which a graph G is said to be a cordial graph if there is a vertex labeling $f: V \to \{0,1\}$ such that it induces an edge labeling $f: E \to \{0,1\}$, which is defined by $f^*(uv) = |f(u) - f(v)|$, satisfies $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_{f^*}(0) - e_{f^*}(1)\right| \leq 1$ (Inayah et al., 2022).

Cordial labeling has experienced significant development and has many types, including product cordial labeling. In product cordial labeling, $f^*(uv) = |f(u) - f(v)|$ is replaced by

$f^*(uv) = f(u)f(v)$ (Inayah et al., 2022). There have been many studies related to product cordial labeling on a graph, some of them being "Product Cordial Labeling of Graphs" (Sundaram et.al., 2004). "Total Product Cordial Labeling of Graphs" (Sundaram et.al., 2005). "Signed Product Cordial Labeling of Some Pan Graphs" (Sadawarte and Srivastav, 2022), "Total Product and Total Edge Product Cordial Labelings of Dragonfly Graph (Dg_n) " (Inayah et.al. 2022).

Based on this study, the author is interested in writing a paper and discussing further the total product cordial labeling on the generalized dragonfly graph $(Dg_n^{(m,k)})$. This paper is based on a paper written by (Inayah et.al. 2022), which discusses the total product cordial labeling and total edge product cordial labeling on dragonfly graphs in general and the total product cordial labeling on dragonfly graphs (Dg_n) in general and the total product cordial labeling on some generalized dragonfly graphs $(Dg_n^{(m,k)})$. So, in this paper, the author proves the total product cordial labeling formula on the generalized dragonfly graphs. $(Dg_n^{(m,k)})$ in general.

DISCUSSION

Definition 2.1 *A graph G is said to be a product cordial graph if there exists a vertex labeling $f: V(G) \to \{0,1\}$ that induces an edge labeling $f^*: E(G) \to \{0,1\}$, which is defined by $f^*(uv) = f(u)f(v)$ and satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$.*

Example 2.1 Given the cycle graph $C_4^{(4)}$ labeled with cordial product labeling as shown in Figure 1.

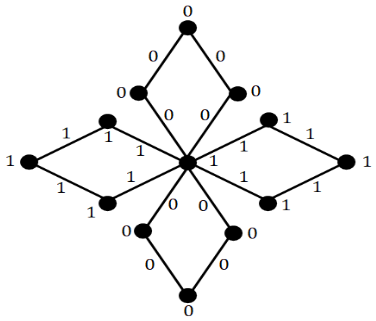


Figure 1. Product cordial labeling of $C_4^{(4)}$

Based on Figure 1, it is known $v_f(0) = 6$, $v_f(1) = 7$, $e_{f^*}(0) = 8$, and $e_{f^*}(1) = 8$. This can prove that the labeling satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. Thus, graph $C_4^{(4)}$ is a product cordial graph.

Definition 2.2 [2] *A graph G is said to be a total product cordial graph if there exists a vertex labeling $f: V(G) \to \{0,1\}$ that induces an edge labeling $f^*: E(G) \to \{0,1\}$, which is defined by $f^*(uv) = f(u)f(v)$ and satisfies* $|(v_f(0) + e_{f^*}(0) - (v_f(1) - e_{f^*}(1))| \leq 1$.

Example 2.2 Consider Example 2.1, the cycle graph $C_4^{(4)}$ with the vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_{13}\}$ and the edge set $E(G) = \{e_1, e_2, e_3, \dots, e_{16}\}$ is a cordial product graph because it satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ with $v_f(0) = 6$, $v_f(1) = 7$, $e_{f^*}(0) = 8$, dan $e_{f^*}(1) = 8$. This proves that graph $C_4^{(4)}$ is also a total cordial product graph because it satisfies $|(v_f(0) + e_{f^*}(0) - (v_f(1) - e_{f^*}(1))| \leq 1$.

Definition 2.3 [2] *For every $m \geq 2$ and $k \geq 1$, the generalized dragonfly graph denoted by $Dg_n^{(m,k)}$ is a graph with vertex set $V(Dg_n^{(m,k)})$ and set of edges $E(Dg_n^{(m,k)})$ as follows:*

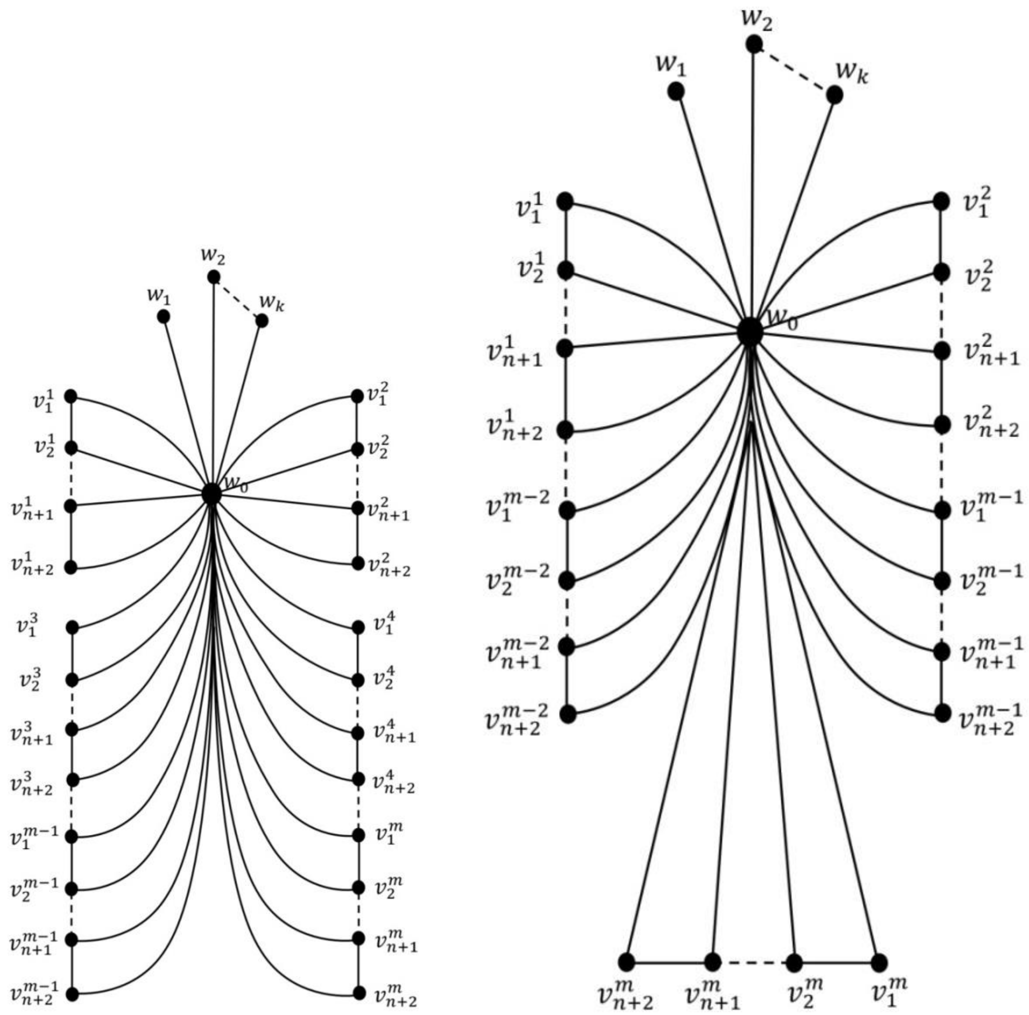
$$V(Dg_n^{(m,k)}) = \{v_i^1, v_i^2, \dots, v_i^m, w_j | \text{ for } i \in \{1,2,\dots,n+2\}, j \in \{0,1,2,\dots,k\}\},$$

$$E(Dg_n^{(m,k)}) = \{v_i^l v_{i+1}^l | \text{ for } i \in \{1,2,\dots,n+1\}, l \in \{1,2,\dots,m\}\} \cup$$

$$\{v_i^l w_0 | \text{ for } i \in \{1,2,\dots,n+2\}, l \in \{1,2,\dots,m\}\} \cup$$

$$\{w_0 w_j | \; j \in \{1,2,\dots,k\}\}.$$

Based on Definition 2.3, a representation of the generalized dragonfly graph ($\boldsymbol{Dg_n^{(m,k)}}$) is given in Figure 2.



(a) (b)

Figure 2. Generalized dragonfly graph ($Dg_n^{(m,k)}$) for (a) m even dan (b) m odd

Theorem 2.1 *For integers $n \geq 2$, $k \geq 1$, and $m \geq 2$ even, the generalized dragonfly graph $Dg_n^{(m,k)}$ is a product cordial graph.*

Proof. Given a generalized dragonfly graph $Dg_n^{(m,k)}$. Suppose n, k is a positive integer and m is an even with $m, n \geq 2$ and $k \geq 1$. Define the function $f: V(Dg_n^{(m,k)}) \to \{0,1\}$, then the following vertex labeling is

$$f(w_0) = 1,$$

$$f(w_i) = 0, \text{for } 1 \le i \le \left\lceil \frac{k}{2} \right\rceil,$$

$$f(w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 1 \le i \le k,$$

$$f(v_i^j) = 1, \text{for } 1 \le i \le n+2, 1 \le j \le \frac{m}{2}$$

$$f(v_i^j) = 0, \text{for } 1 \le i \le n+2, \frac{m}{2} + 1 \le j \le m.$$

Based on the vertex labeling above, we can know that $v_f(0) = \frac{m}{2}(n+2) + \left\lceil \frac{k}{2} \right\rceil$ and $v_f(1) = \frac{m}{2}(n+2) + \left\lfloor \frac{k}{2} \right\rfloor + 1$. Next, we can know the edge labeling $f^*: E(Dg_n^{(m,k)}) \to \{0,1\}$ as follows

$$f(w_0 w_i) = 0, \text{for } 1 \le i \le \left\lceil \frac{k}{2} \right\rceil,$$

$$f(w_0 w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 1 \le i \le k,$$

$$f(v_i^j w_0) = 0, \text{for } 1 \le i \le n+2, \frac{m}{2} + 1 \le j \le m,$$

$$f(v_i^j v_{i+1}^j) = 0, \text{for } 1 \le i \le n+1, \frac{m}{2} + 1 \le j \le m$$

$$f(v_i^1 w_0) = 1, \text{for } 1 \le i \le n+2, 1 \le j \le \frac{m}{2},$$

$$f(v_i^j v_{i+1}^j) = 1, \text{for } 1 \le i \le n+1, 1 \le j \le \frac{m}{2}.$$

Based on the edge labeling above, we can know $e_{f^*}(0) = \frac{m}{2}(2n+3) + \left\lceil \frac{k}{2} \right\rceil$ and $e_{f^*}(1) = \frac{m}{2}(2n+3) + \left\lfloor \frac{k}{2} \right\rfloor$. This proves $|v_f(0) - v_f(1)| \le 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \le 1$. So, it is proven that the f is a product cordial labeling for the generalized dragonfly graphs $(Dg_n^{(m,k)})$ with $n \ge 2$, $k \ge 1$ positive integers, and $m \ge 2$ even. ■

Corollary 2.1 *For integers $n \ge 2$, $k \ge 1$, and $m \ge 2$ even, the generalized dragonfly graph $(Dg_n^{(m,k)})$ is a total product cordial graph.*

Proof: By using Theorem 2.1, it can be shown that the labeling satisfies $\left|(e_{f^*}(0)+v_f(0))-(e_{f^*}(1)+v_f(1))\right|\leq 1$. This proves that the generalized dragonfly graph $(Dg_n^{(m,k)})$ for integers $n\geq 2, k\geq 1$, and $m\geq 2$ even is a total product cordial graph. ■

Example 2.3 Given a generalized dragonfly graph $Dg_3^{(2,1)}$, by using Theorem 2.1, we obtain a vertex labeling $f: V(Dg_3^{(2,1)})\to\{0,1\}$ which induces the edge labeling $f^*: E(Dg_3^{(2,1)})\to\{0,1\}$ in Figure 3 as follows.

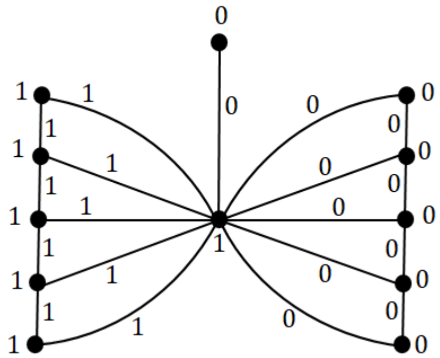


Figure 3. Total product cordial labeling of $Dg_3^{(2,1)}$ graph.

Based on Figure 3, it is known that $v_f(0)=6$, $v_f(1)=6$, $e_{f^*}(0)=10$, and $e_{f^*}(1)=9$. This proves that the $Dg_3^{(2,1)}$ graph is a total product cordial graph because it satisfies $\left|(e_{f^*}(0)+v_f(0))-(e_{f^*}(1)+v_f(1))\right|\leq 1$.

Theorem 2.2 *Given a generalized dragonfly graph* $\boldsymbol{Dg_n^{(m,k)}}$*, for integers* $\boldsymbol{k,n,m\geq 2}$ *with* $\boldsymbol{n}$ *even* $(\boldsymbol{n=2a})$ *and* $\boldsymbol{m}$ *odd, then* $\boldsymbol{Dg_{2a}^{(m,k)}}$ *is a product cordial graph.*

Proof: Suppose $Dg_{2a}^{(m,k)}$ is a generalized dragonfly graph with $a\geq 1$, $k,m\geq 2$, and m odd. Define the function $f: V\left(Dg_{2a}^{(m,k)}\right)\to\{0,1\}$, then there are the following two cases.

Case 1. Let k be even. The vertex labeling of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$ is

$f(w_0)=1,$

$f(w_i)=0, \text{for } 1\leq i\leq \frac{k}{2},$

$$f(w_i) = 1, \text{for } \frac{k}{2} + 1 \le i \le k,$$

$$f(v_i^j) = 1, \text{for } 1 \le i \le 2a + 2, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor$$

$$f(v_i^j) = 0, \text{for } 1 \le i \le 2a + 2, \left\lceil \frac{m}{2} \right\rceil \le j \le m - 1$$

$$f(v_i^m) = 0, \text{for } 1 \le i \le a + 1,$$

$$f(v_i^m) = 1, \text{for } a + 2 \le i \le 2a + 2.$$

Based on the vertex labeling above, we know $v_f(0) = m(a+1) + \frac{k}{2}$ and $v_f(1) = m(a+1) + \frac{k}{2} + 1$. So, the edge of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$, which is labeled with zero is

$$f(w_0 w_i) = 0, \text{for } 1 \le i \le \frac{k}{2},$$

$$f(v_i^j w_0) = 0, \text{for } 1 \le i \le 2a + 2, \left\lceil \frac{m}{2} \right\rceil \le j \le m - 1,$$

$$f(v_i^m w_0) = 0, \text{for } 1 \le i \le a + 1,$$

$$f(v_i^j v_{i+1}^j) = 0, \text{for } 1 \le i \le 2a + 1, \left\lceil \frac{m}{2} \right\rceil \le j \le m - 1,$$

$$f(v_i^m v_{i+1}^m) = 0, \text{for } 1 \le i \le a + 1,$$

while the edge of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$, which is labeled with one is

$$f(w_0 w_i) = 1, \text{for } \frac{k}{2} + 1 \le i \le k,$$

$$f(v_i^j w_0) = 1, \text{for } 1 \le i \le 2a + 2, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f(v_i^m w_0) = 1, \text{for } a + 2 \le i \le 2a + 2,$$

$$f(v_i^j v_{i+1}^j) = 1, \text{for } 1 \le i \le 2a + 1, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f(v_i^m v_{i+1}^m) = 1, \text{for } a + 2 \le i \le 2a + 1.$$

Based on the edge labeling above, we know $e_{f^*}(0) = 2m(a+1) + \frac{k}{2} - \left\lfloor \frac{m}{2} \right\rfloor$ and $e_{f^*}(1) = 2m(a+1) + \frac{k}{2} - \left\lceil \frac{m}{2} \right\rceil$. This proves $\left|v_f(0) - v_f(1)\right| \le 1$ and $\left|e_{f^*}(0) - e_{f^*}(1)\right| \le 1$. So, it is proven that the f is a product cordial labeling for the generalized dragonfly graph $Dg_{2a}^{(m,k)}$ with $m, k \geq 2$, $a \geq 1$, and m odd.

Case 2. Let k be odd. The vertex labeling of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$ is

$$f(w_0) = 1,$$

$$f(w_i) = 0, \text{for } 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil + 1,$$

$$f(w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 2 \leq i \leq k,$$

$$f(v_i^j) = 1, \text{for } 1 \leq i \leq 2a + 2, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f(v_i^j) = 0, \text{for } 1 \leq i \leq 2a + 2, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m - 1,$$

$$f(v_i^m) = 0, \text{for } 1 \leq i \leq a,$$

$$f(v_i^m) = 1, \text{for } a + 1 \leq i \leq 2a + 2.$$

Based on the vertex labeling above, we know $v_f(0) = m(a+1) + \left\lceil \frac{k}{2} \right\rceil$ and $v_f(1) = m(a+1) + \left\lfloor \frac{k}{2} \right\rfloor + 1$. So, the edge of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$, which is labeled with zero is

$$f(w_0 w_i) = 0, \text{for } 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil + 1,$$

$$f(v_i^j w_0) = 0, \text{for } 1 \leq i \leq 2a + 2, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m - 1,$$

$$f(v_i^m w_0) = 0, \text{for } 1 \leq i \leq a,$$

$$f(v_i^j v_{i+1}^j) = 0, \text{for } 1 \leq i \leq 2a + 1, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m - 1,$$

$$f(v_i^m v_{i+1}^m) = 0, \text{for } 1 \leq i \leq a,$$

while the edge of the generalized dragonfly graph $Dg_{2a}^{(m,k)}$, which is labeled with one is

$$f(w_0 w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 2 \leq i \leq k,$$

$$f(v_i^j w_0) = 1, \text{for } 1 \leq i \leq 2a + 2, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

$f(v_i^m w_0) = 1,$ for $a+1 \le i \le 2a+2,$

$f\left(v_i^j v_{i+1}^j\right) = 1,$ for $1 \le i \le 2a+1, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$

$f(v_i^m v_{i+1}^m) = 1,$ for $a+1 \le i \le 2a+1.$

Based on the edge labeling above, we know $e_{f^*}(0) = 2m(a+1) + \left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{m}{2} \right\rceil$ and $e_{f^*}(1) = 2m(a+1) + \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor$. This proves $|v_f(0) - v_f(1)| \le 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \le 1$. So, it is proven that the f is a product cordial labeling for the generalized dragonfly graph $Dg_{2a}^{(m,k)}$ with $k, m \ge 2$ and $a \ge 1$. Therefore, considering these two cases, it can be proven that the generalized of dragonfly graphs $Dg_n^{(m,k)}$ for integers $k, m, n \ge 2$ and n even is a product cordial graph. ■

Corollary 2.2 *For integers* $\boldsymbol{k, n, m \ge 2}$ *with* $\boldsymbol{n}$ *even and* $\boldsymbol{m}$ *odd, then* $\boldsymbol{Dg_n^{(m,k)}}$ *is a total product cordial graph.*

Proof: By using Theorem 2.2, it can be shown that the labeling satisfies $\left|(e_{f^*}(0) + v_f(0)) - (e_{f^*}(1) + v_f(1))\right| \le 1$. This proves the generalized dragonfly graph $Dg_n^{(m,k)}$ for integers $k, n, m \ge 2$ with n even and m odd is a total product cordial graph. ■

Example 2.4 Given a generalized dragonfly graph $Dg_2^{(3,2)}$, by using Case 1 in Theorem 2.2, we obtain a vertex labeling $f: V(Dg_2^{(3,2)}) \rightarrow \{0,1\}$ which induces the edge labeling $f^*: E(Dg_2^{(3,2)}) \rightarrow \{0,1\}$ in Figure 4 as follows:

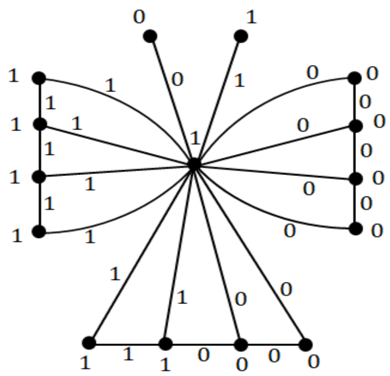


Figure 4. Total product cordial labeling of $Dg_2^{(3,2)}$ graph

Based on Figure 4, it is known that $v_f(0) = 7$, $v_f(1) = 7$, $e_{f^*}(0) = 12$, and $e_{f^*}(1) = 11$. This proves that the graph $Dg_2^{(3,2)}$ is a total product cordial graph because it satisfies $\left|(e_{f^*}(0) + v_f(0)) - (e_{f^*}(1) + v_f(1))\right| \leq 1$.

Teorema 2.3 *Given a generalized dragonfly graph $Dg_n^{(m,k)}$, for integers $k \geq 1$ and $n, m \geq 2$ with n, m odd $(n = 2a + 1)$ then $Dg_{2a+1}^{(m,k)}$ is a product cordial graph*

Proof: Suppose $Dg_{2a+1}^{(m,k)}$ is a generalized dragonfly graph with $a, k \geq 1$, $m \geq 2$ integers and m odd. Define the function $f: V\left(Dg_{2a+1}^{(m,k)}\right) \to \{0,1\}$, and then there are two cases.

Case 1. Let k be odd. The vertex labeling of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$ is

$$f(w_0) = 1,$$

$$f(w_i) = 0, \text{for } 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$f(w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 1 \leq i \leq k,$$

$$f\left(v_i^j\right) = 1, \text{for } 1 \leq i \leq 2a + 3, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f\left(v_i^j\right) = 0, \text{for } 1 \leq i \leq 2a + 3, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m - 1,$$

$$f(v_i^m) = 0, \text{for } 1 \leq i \leq a + 1,$$

$$f(v_i^m) = 1, \text{for } a + 2 \leq i \leq 2a + 3.$$

Based on the vertex labeling above, we know $v_f(0) = m(a + 1) + \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor$ and $v_f(1) = m(a + 1) + \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil + 1$. So, the edge of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$, which is labeled with zero is

$$f(w_0 w_i) = 0, \text{for } 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$f\left(v_i^j w_0\right) = 0, \text{for } 1 \leq i \leq 2a + 3, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m - 1$$

$$f(v_i^m w_0) = 0, \text{for } 1 \leq i \leq a + 1,$$

$f\left(v_i^j v_{i+1}^j\right) = 0, \text{for } 1 \leq i \leq 2a+2, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m-1,$

$f(v_i^m v_{i+1}^m) = 0, \text{for } 1 \leq i \leq a+1,$

while the edge of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$, which is labeled with one is

$f(w_0 w_i) = 1, \text{for } \left\lceil \frac{k}{2} \right\rceil + 1 \leq i \leq k,$

$f\left(v_i^j w_0\right) = 1, \text{for } 1 \leq i \leq 2a+3, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$

$f(v_i^m w_0) = 1, \text{for } a+2 \leq i \leq 2a+3,$

$f\left(v_i^j v_{i+1}^j\right) = 1, \text{for } 1 \leq i \leq 2a+2, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$

$f(v_i^m v_{i+1}^m) = 1, \text{for } a+2 \leq i \leq 2a+2.$

Based on the edge labeling above, we know $e_f(0) = 2m(a+1) + \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil$ and $e_f(1) = 2m(a+1) + \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil$. This proves $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_{f^*}(0) - e_{f^*}(1)\right| \leq 1$. So, it is proven that the f is a product cordial labeling for the generalized of dragonfly graphs $Dg_{2a+1}^{(m,k)}$ with integers $a, k \geq 1$, $m \geq 2$, and m odd.

Case 2. Let k be even. The vertex labeling of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$ is

$f(w_0) = 1,$

$f(w_i) = 0, \text{for } 1 \leq i \leq \frac{k}{2} + 1,$

$f(w_i) = 1, \text{for } \frac{k}{2} + 2 \leq i \leq k,$

$f\left(v_i^j\right) = 1, \text{for } 1 \leq i \leq 2a+3, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$

$f\left(v_i^j\right) = 0, \text{for } 1 \leq i \leq 2a+3, \left\lceil \frac{m}{2} \right\rceil \leq j \leq m-1,$

$f(v_i^m) = 0, \text{for } 1 \leq i \leq a+1,$

$f(v_i^m) = 1, \text{for } a+2 \leq i \leq 2a+3.$

Based on the vertex labeling above, we know $v_f(0) = m(a+1) + \frac{k}{2} + \left\lceil \frac{m}{2} \right\rceil$ and $v_f(1) = m(a+1) + \frac{k}{2} + \left\lceil \frac{m}{2} \right\rceil$. So, the edge of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$, which is labeled with zero is

$$f(w_0 w_i) = 0, \text{for } 1 \le i \le \frac{k}{2} + 1,$$

$$f\left(v_i^j w_0\right) = 0, \text{for } 1 \le i \le 2a+3, \left\lceil \frac{m}{2} \right\rceil \le j \le m-1,$$

$$f(v_i^7 w_0) = 0, \text{for } 1 \le i \le a+1,$$

$$f\left(v_i^j v_{i+1}^j\right) = 0, \text{for } 1 \le i \le 2a+2, \left\lceil \frac{m}{2} \right\rceil \le j \le m-1,$$

$$f(v_i^m v_{i+1}^m) = 0, \text{for } 1 \le i \le a+1,$$

while the edge of the generalized dragonfly graph $Dg_{2a+1}^{(m,k)}$, which is labeled with one is

$$f(w_0 w_i) = 1, \text{for } \frac{k}{2} + 2 \le i \le k,$$

$$f\left(v_i^j w_0\right) = 1, \text{for } 1 \le i \le 2a+3, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f(v_i^m w_0) = 1, \text{for } a+2 \le i \le 2a+3,$$

$$f\left(v_i^j v_{i+1}^j\right) = 1, \text{for } 1 \le i \le 2a+2, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$f(v_i^m v_{i+1}^m) = 1, \text{for } a+2 \le i \le 2a+2.$$

Based on the edge labeling above, we know $e_f(0) = 2m(a+1) + \frac{k}{2} + \left\lceil \frac{m}{2} \right\rceil$ and $e_f(1) = 2m(a+1) + \frac{k}{2} + \left\lfloor \frac{m}{2} \right\rfloor$. This proves $\left|v_f(0) - v_f(1)\right| \le 1$ and $\left|e_{f^*}(0) - e_{f^*}(1)\right| \le 1$. So, it is proven that the f is a product cordial labeling for the generalized dragonfly graphs $Dg_{2a+1}^{(m,k)}$ with integers $a, k \ge 1$, $m \ge 2$ and m odd. Therefore, considering these two cases, it can be proven that the generalized dragonfly graphs $Dg_n^{(m,k)}$ for integers $k \ge 1$, $m, n \ge 2$, and m, n is product cordial graph. ■

Corollary 2.3 *For integers* $\boldsymbol{k \ge 1}$, $\boldsymbol{m, n \ge 2}$ *and* $\boldsymbol{m, n}$ *odd, then* $\boldsymbol{Dg_n^{(m,k)}}$ *is a total product cordial graph.*

Proof: By using Theorem 2.3, it can be shown that the labeling satisfies $|(e_{f^*}(0)+v_f(0))-(e_{f^*}(1)+v_f(1))|\leq 1$. This proves the generalized dragonfly graph $Dg_n^{(m,k)}$ for integers $k\geq 1$, $m,n\geq 2$ and m,n is a total product cordial graph. ■

Example 2.5. Given a generalized dragonfly graph $Dg_3^{(3,2)}$, by using Case 2 in Theorem 2.3, we obtain a vertex labeling $f: V(Dg_3^{(3,2)})\rightarrow\{0,1\}$ which induces the edge labeling $f^*: E(Dg_3^{(3,2)})\rightarrow\{0,1\}$ in Figure 5 as follows:

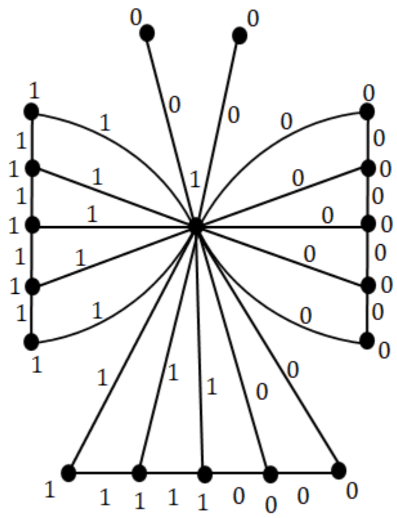


Figure 5. Total product cordial labeling of $Dg_3^{(3,2)}$ graph

Based on Figure 5, it is known that $v_f(0)=9$, $v_f(1)=9$, $e_{f^*}(0)=15$, and $e_{f^*}(1)=14$. This proves that the graph $Dg_3^{(3,2)}$ is a total product cordial graph because it satisfies $|(e_{f^*}(0)+v_f(0))-(e_{f^*}(1)+v_f(1))|\leq 1$.

CONCLUSIONS

Based on Theorems 2.1, 2.2, 2.3, and Corollary 2.1, 2.2, and 2.3, the generalized dragonfly graphs ($Dg_n^{(m,k)}$) with $m,n\geq 2$, and $k\geq 1$ integers are a total product cordial graph except for $Dg_n^{(m,1)}$ with m odd and n even.

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