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Efficiency Measures of a Class of Regular-Graph Semi-Latin Rectangles with Block Size Two and Odd Number of Treatments

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Abstract: *A semi-Latin rectangle (SLR), denoted (h × p)/k, is a row-column design consisting of v treatments in h rows and p columns (where h may or may not be equal to p), where each row-column intersection (cell) which constitutes a block contains k treatments and each treatment of the design occurs the same number of times in each row, denoted* n_r *and the same number of times in each column, denoted* n_c *(where* n_r *and* n_c *need not be all 1). Regular-graph semi-Latin rectangles (RGSLRs) are SLRs which possess the property that, for any two pairs of distinct treatments, their concurrences differ by at most one. This work considers RGSLRs with k = 2, where the number of treatments (v) is odd and h = p = v, which are the smallest RGSLRs for odd values of v. Construction for such design is given in Uto and Bailey (2022). However, the foregoing paper does not give the efficiency measures of these designs. We determine the A-, D- and E-efficiency measures of these designs for some odd values of v to give information on how good the designs are for experimentation. Results show that the designs have good efficiency measures, hence are good designs for experiments involving designs of their sizes.*

Keywords: Semi-Latin rectangle, Regular-graph semi-Latin rectangle, Quotient block design, Canonical efficiency factor, Connected design, Efficiency measures.

INTRODUCTION

In a semi-Latin rectangle (SLR), there are *v* treatments displayed in *h* rows and *p* columns with *k* treatments in each row-column intersection (block). Moreover, each treatment occurs n_r and

n_c times in each row and each column, respectively. Such a design is denoted $(h \times p)/k$, where h may (or may not) be equal to p. If $k = 2$, as in this work, the SLR is said to have block size two. A SLR, being a row-column design, generalizes the Latin square (LS) and semi-Latin square (SLS), and are useful designs in conducting experiments in agriculture, industry, amongst others: see Bailey and Monod (2001), Uto and Bailey (2020, 2022), and Yadav *et al.* (2024). Discussions on LS can be found in papers such as Bose and Nair (1941), keedwell and Mullen (2004). Similarly, discussions on SLS can be found in papers such as Preece and Freeman (1983), Bailey (1988, 1992), Bedford and Whitaker (2001), Chigbu (2003) and Soicher (2013).

If the rows and columns of a SLR are ignored, then the resulting design is its quotient block design (QBD). If the QBD of a SLR is a regular-graph design (RGD), that is, has the property that, for any two pairs of distinct treatments, their concurrences differ by at most one, then the SLR is known as a regular-graph semi-Latin rectangle (RGSLR): see Bailey and Monod (2001) and Uto and Bailey (2022). RGSLRs contain optimal design for experiments in situation where a balanced semi-Latin rectangle (BSLR), which is known to be *A*-, *D*- and *E*-optimal fails to exist: see Uto and Bailey (2022). For discussions on BSLR, see Uto and Bailey (2020) and Yadav *et al.* (2024). Some common measures of the efficiency of a SLR are the *A*-, *D*- and *E*-efficiency measures, which are applied to its QBD.

Bailey and Monod (2001) provided constructions for efficient SLRs having n rows, $2n$ columns, $2n$ treatments and block size two, where n lies between 2 and 10, inclusive. Starter and balanced tournament design were used in constructing the designs. Uto and Bailey (2020) found a subclass of SLRs known as the BSLR and gave the combinatorial properties, necessary conditions for its existence, and algorithms for the construction of the design when $k = 2$ for cases that v is odd and even, respectively. Uto and Bailey (2022) developed constructions for RGSLRs with $k = 2$. When v is odd, the construction employs bi-starter to obtain the smallest design. Constructions for larger RGSLRs are also given for each case, when v is odd and when it is even.Yadav *et al.* (2024) gave an algorithm for constructing BSLRs with $k = 4$. The paper also gives the average efficiency factor for each BSLR obtainable for v in the range $5 \le v \le 20$ .This work determines the efficiency measures of the smallest designs for certain odd values of v whose construction is given in Uto and Bailey (2022). These designs have block size, k = 2 and are restricted to $h = p = v$. They are precisely $(v \times v)/2$ RGSLRs for v treatments.

Preliminaries

Some useful matrices

Some matrices that will be useful in this work include the incidence matrix, concurrence matrix and the information matrix, particularly, the scaled information matrix. We give a description of these matrices.

i Incidence matrix

The incidence matrix $N = (n_{ij})$ of a design is a $v \times b$ matrix whose (i, j)th entry is the number of times that treatment i appears in block j, where v denotes the number of treatments and b denotes the number of blocks. In particular, for the class of designs under consideration in this work, $b = v^2$.

It is noted here that if $n_{ij} = 0$ or 1 $\forall\,(i, j)$, where $i = 1, \ldots, v$ and $j = 1, \ldots, b$, then the design is said to be binary, that is, no treatment appears more than once in any block, which is typical of a SLR.

ii Concurrence Matrix

The concurrence matrix, NN' of a design is a $v \times v$ matrix whose (i, j)th entry is the number of concurrences, λ_{ij} between treatments i and j, that is ,the number of times that treatment i appears together with treatment j.

We observe that for binary designs, the $(i, j)th$ entry of NN' is identical to the number of blocks in which treatment i appears together with treatment j. Moreover, if the design is binary, then the i^{th} leading diagonal entry of its concurrence matrix is identical to r_i, the number of replications of treatment i.

iii Information Matrix

The information matrix, L of a design is a $v \times v$ matrix defined by $L = rI - \frac{1}{k} NN'$, where I is a $v \times v$ identity matrix, r and k are parameters denoting the number of replications of each treatment and the size of each block, respectively. We note that, in this work, $r = 2v$.

Some properties of the information matrix:

1. The rank of L denoted by rank $(L) \leq v-1$, In particular, if rank $(L) = v-1$, then the design whose information matrix is L is a connected design, otherwise it is a disconnected design.
2. L is symmetric and has a full set of orthogonal eigenvectors
3. Let $\underline{a}' = (1\ 1\ \ .\ .\ .\ \ 1)$, that is, $\underline{a}'$ is a row vector of ones (1's). Then $L\,\underline{a} = \underline{0}$. So L has an eigenvalue 0. In a connected design, the multiplicity of 0 is 1. However in a disconnected design, the multiplicity of 0 is more than 1.

iv Scaled information matrix

Let $L^* = I - \left(\frac{1}{rk}\right) NN'$. Then L^* is called the scaled information matrix of a design: see Soicher, 2013.

Remark

If D denote a connected design, that is, a design in which all the elementary (simple) contrasts of treatments are estimable, and L^* denote the scaled information matrix of D. Denote by e_i, $i = 1, \ldots, v$, the eigenvalues of L^*. Suppose $e_v = 0$. Then e_i, $i = 1, \ldots, v - 1$ are called the canonical efficiency factors (c. e. f_s) of D: see, for example, Chigbu (1998, Chapter 4). Hence, if D is a connected design, then each non-zero eigenvalue of L^* corresponding to D is called a canonical efficiency factor (c. e. f), e_i, where $i = 1, \ldots, v - 1$. Moreover, the eigenvalue, zero corresponds to an all-one eigenvector: see, for example, Chigbu *et al.* (2002, Chapter 3).

The *A*-, *D*- and *E*-efficiency measures of a design

The A- efficiency measure of a design is given in terms of its canonical efficiency factors as

$$A = \frac{v-1}{\sum_{i=1}^{v-1} \frac{1}{e_i}}$$

where v is the number of treatments and $e_1, e_2, \ldots, e_{v-1}$ are the canonical efficiency factors (c. e. f_s) of the design.

Similarly, the D-efficiency measure of a design is given in terms of its canonical efficiency factors as

$$D = \left(\prod_{i=1}^{v-1} e_i \right)^{\frac{1}{v-1}}$$

Moreover, the E- efficiency measure of a design is given in terms of its canonical efficiency factors as

$E = \min(e_1, e_2, \ldots, e_{v-1})$.

We note that, if a design has the highest value of an efficiency measure over all possible realizable designs in its class, then it is said to be optimal with respect to an optimality criterion corresponding to that efficiency measure. For instance, if a design has the highest value of the A-efficiency measure over its class, it is said to be A-optimal. D-optimal and E-optimal designs are similarly defined. For a discussion of these efficiency measures and how they are linked to optimality of designs: see, for example, Bailey and Royle (1997) and Chigbu (1998, Chapter 5).

METHODS

We begin by generating the $(v \times v)/2$ RGSLRs for $v = 7$, 9 and 11 by employing the algorithm given in Uto and Bailey (2022). However, for $v = 5$, the corresponding design is given as an example in Fig.1 of Uto and Bailey (2022). With the aid of MATLAB, the incidence matrix, and scaled information matrix were generated for each RGSLR under consideration via its QBD, and the c. e. f_s of each design were also obtained. Subsequently, the *A*-, *D*- and *E*-efficiency measures were computed for each RGSLR. These efficiency measures are the commonly used efficiency measures of a design: see, Bailey and Royle (1997).

RESULTS

The RGSLRs whose efficiency measures are sought

The (5×5)/2 RGSLR for 5 treatments presented in Figure 1 of this work is given as an example in Fig. 1 of Uto and Bailey (2022).

1 5	5 2	2 4	4 3	3 1
2 1	1 3	3 5	5 4	4 2
3 2	2 4	4 1	1 5	5 3
4 3	3 5	5 2	2 1	1 4
5 4	4 1	1 3	3 2	2 5

Figure 1: The (5×5)/2 RGSLR for 5 treatments, Λ_1.

For $v = 7$, 9 and 11, the corresponding designs, obtained by employing the algorithm presented in Section 3 of Uto and Bailey (2022), are presented in Figures 2, 3, and 4, respectively.

1 7	7 2	2 6	6 3	3 5	5 4	4 1
2 1	1 3	3 7	7 4	4 6	6 5	5 2
3 2	2 4	4 1	1 5	5 7	7 6	6 3
4 3	3 5	5 2	2 6	6 1	1 7	7 4
5 4	4 6	6 3	3 7	7 2	2 1	1 5
6 5	5 7	7 4	4 1	1 3	3 2	2 6
7 6	6 1	1 5	5 2	2 4	4 3	3 7

Figure 2: The (7×7)/2 RGSLR for 7 treatments, Λ_2.

1 9	9 2	2 8	8 3	3 7	7 4	4 6	6 5	5 1
2 1	1 3	3 9	9 4	4 8	8 5	5 7	7 9	6 2
3 2	2 4	4 1	1 5	5 9	9 6	6 8	8 7	7 3
4 3	3 5	5 2	2 6	6 1	1 7	7 9	9 8	8 4
5 4	4 6	6 3	3 7	7 2	2 8	8 1	1 9	9 5
6 5	5 7	7 4	4 8	8 3	3 9	9 2	2 1	1 6
7 6	6 8	8 5	5 9	9 4	4 1	1 3	3 2	2 7
8 7	7 9	9 6	6 1	1 5	5 2	2 4	4 3	3 8
9 8	8 1	1 7	7 2	2 6	6 3	3 5	5 4	4 9

Figure 3: The (9×9)/2 RGSLR for 9 treatments, Λ_3.

1 11	11 2	2 10	10 3	3 9	9 4	4 8	8 5	5 7	7 6	6 1
2 1	1 3	3 11	11 4	4 10	10 5	5 9	9 6	6 8	8 7	7 2
3 2	2 4	4 1	1 5	5 11	11 6	6 10	10 7	7 9	9 8	8 3
4 3	3 5	5 2	2 6	6 1	1 7	7 11	11 8	8 10	10 9	9 4
5 4	4 6	6 3	3 7	7 2	2 8	8 1	1 9	9 11	11 10	10 5
6 5	5 7	7 4	4 8	8 3	3 9	9 2	2 10	10 1	1 11	11 6
7 6	6 8	8 5	5 9	9 4	4 10	10 3	3 11	11 2	2 1	1 7
8 7	7 9	9 6	6 10	10 5	5 11	11 4	4 1	1 3	3 2	2 8
9 8	8 10	10 7	7 11	11 6	6 1	1 5	5 2	2 4	4 3	3 9
10 9	9 11	11 8	8 1	1 7	7 2	2 6	6 3	3 5	5 4	4 10
11 10	10 1	1 9	9 2	2 8	8 3	3 7	7 4	4 6	6 5	5 11

Figure 4: The (11×11)/2 RGSLR for 11 treatments, Λ_4.

The incidence and scaled information matrices of the designs

The incidence matrices, N_1, N_2, N_3 and N_4 of the QBDs of Λ_1, Λ_2, Λ_3 and Λ_4 are presented in Tables 1, 2, 3 and 4, respectively.

Table 1: The incidence matrix, N_1

Columns 1 through 21

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Columns 22 through 25

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Table 2: The incidence matrix, N_2

Columns 1 through 26

$$\begin{pmatrix}
1&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0\\
0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0\\
0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&1\\
0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&1&0&1&1&0&0&0&1&0&0&1&0&0&1&0\\
1&0&0&0&0&0&1&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1
\end{pmatrix}$$

Columns 27 through 49

$$\begin{pmatrix}
1&0&0&0&0&1&0&1&0&0&0&0&1&1&0&0&1&0&0&0&1&0&0\\
0&1&0&0&0&0&1&0&1&0&0&0&0&1&1&0&0&1&0&0&0&1&0\\
0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&1\\
1&0&0&1&0&0&0&0&1&1&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&1&1&0&1&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&1&0&1&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&1&0&1&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&1
\end{pmatrix}$$

Table 3: The incidence matrix, N_3

Columns 1 through 21

$$\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1\\
0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&1&0&0\\
1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&1&0
\end{pmatrix}$$

Columns 22 through 42

0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0
0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	1	0	0	0	1	0
0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0

Columns 43 through 63

0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0
0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1
1	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0
0	1	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1
0	0	1	0	0	1	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0
1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0

Columns 64 through 81

0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0	0
0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0
0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1
1	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	1

Table 4: The incidence matrix, N_4

Columns 1 through Columns 21

1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1

Columns 22 through Columns 42

1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1

Columns 43 through 63

1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

Columns 64 through 84

1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0
0	0	1	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0
0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1

Columns 85 through 105

1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0
0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	1	0
0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	1

Columns 106 through 121

1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
1	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0
0	1	1	0	0	0	0	1	0	0	0	0	0	1	0	0
0	0	1	1	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

The scaled information matrices, $L_1{}^*$, $L_2{}^*$, $L_3{}^*$ and $L_4{}^*$ of Λ_1, Λ_2, Λ_3 and Λ_4 are presented in Tables 5, 6, 7 and 8, respectively.

Table 5: The scaled information matrix, $L_1{}^*$

0.5000	-0.1000	-0.1500	-0.1500	-0.1000
-0.1000	0.5000	-0.1000	-0.1500	-0.1500
-0.1500	-0.1000	0.5000	-0.1000	-0.1500
-0.1500	-0.1500	-0.1000	0.5000	-0.1000
-0.1000	-0.1500	-0.1500	-0.1000	0.5000

Table 6: The scaled information matrix, $L_2{}^*$

0.5000	-0.0714	-0.0714	-0.1071	-0.1071	-0.0714	-0.0714
-0.0714	0.5000	-0.1071	-0.0714	-0.1071	-0.1071	-0.0714
-0.0714	-0.0714	0.5000	-0.0714	-0.0714	-0.1071	-0.1071
-0.1071	-0.0714	-0.0714	0.5000	-0.0714	-0.0714	-0.1071
-0.1071	-0.1071	-0.0714	-0.0714	0.5000	-0.0714	-0.0714
-0.0714	-0.1071	-0.1071	-0.0714	-0.0714	0.5000	-0.0714
-0.0714	-0.0714	-0.1071	-0.1071	-0.0714	-0.0714	0.5000

Table 7: The scaled information matrix, $L_3{}^*$

$$\begin{pmatrix}
0.5000 & -0.0556 & -0.0556 & -0.0556 & -0.0833 & -0.0833 & -0.0556 & -0.0556 & -0.0556 \\
-0.0556 & 0.5000 & -0.0556 & -0.0556 & -0.0556 & -0.0833 & -0.0833 & -0.0556 & -0.0556 \\
-0.0556 & -0.0556 & 0.5000 & -0.0556 & -0.0556 & -0.0556 & -0.0833 & -0.0833 & -0.0556 \\
-0.0556 & -0.0556 & -0.0556 & 0.5000 & -0.0556 & -0.0556 & -0.0556 & -0.0833 & -0.0833 \\
-0.0833 & -0.0556 & -0.0556 & -0.0556 & 0.5000 & -0.0556 & -0.0556 & -0.0556 & -0.0833 \\
-0.0833 & -0.0833 & -0.0556 & -0.0556 & -0.0556 & 0.5000 & -0.0556 & -0.0556 & -0.0556 \\
-0.0556 & -0.0833 & -0.0833 & -0.0556 & -0.0556 & -0.0556 & 0.5000 & -0.0556 & -0.0556 \\
-0.0556 & -0.0556 & -0.0833 & -0.0833 & -0.0556 & -0.0556 & -0.0556 & 0.5000 & -0.0556 \\
-0.0556 & -0.0556 & -0.0556 & -0.0833 & -0.0833 & -0.0556 & -0.0556 & -0.0556 & 0.5000
\end{pmatrix}$$

Table 8: The Scaled information matrix, $L_4{}^*$

$$\begin{pmatrix}
0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 \\
-0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 \\
-0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 \\
-0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 \\
-0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 \\
-0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 \\
-0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 & -0.0455 \\
-0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 & -0.0455 \\
-0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 & -0.0455 \\
-0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000 & -0.0455 \\
-0.0455 & -0.0455 & -0.0455 & -0.0455 & -0.0682 & -0.0682 & -0.0455 & -0.0455 & -0.0455 & -0.0455 & 0.5000
\end{pmatrix}$$

4.3 The c. e. fs and *A*-, *D*- and *E*-efficiency measures of the designs

The eigenvalues of $L_1{}^*$, $L_2{}^*$, $L_3{}^*$ and $L_4{}^*$ are the components of the vectors

$$\begin{pmatrix} 0.0000 \\ 0.5691 \\ 0.5691 \\ 0.6809 \\ 0.6809 \end{pmatrix}, \begin{pmatrix} 0.0000 \\ 0.5269 \\ 0.5269 \\ 0.5873 \\ 0.5873 \\ 0.6358 \\ 0.6358 \end{pmatrix}, \begin{pmatrix} 0.0000 \\ 0.5130 \\ 0.5130 \\ 0.5459 \\ 0.5459 \\ 0.5833 \\ 0.5833 \\ 0.6078 \\ 0.6078 \end{pmatrix} \text{ and } \begin{pmatrix} 0.0000 \\ 0.5072 \\ 0.5072 \\ 0.5266 \\ 0.5266 \\ 0.5519 \\ 0.5519 \\ 0.5752 \\ 0.5752 \\ 0.5891 \\ 0.5891 \end{pmatrix}$$

, respectively, such that the c. e. f_s of Λ_1 are 0.5691 with multiplicity 2, and 0.6809 (also with multiplicity 2). Similarly, the c. e. f_s of Λ_2 are 0.5269 with multiplicity 2, 0.5873 with multiplicity 2, and 0.6358 with multiplicity 2. Moreover, for Λ_3, its c. e. f_s are 0.5130 with multiplicity 2, 0.5459 with multiplicity 2, 0.5833 with multiplicity 2, and 0.6078 with multiplicity 2; and finally, for Λ_4, its c. e. f_s are 0.5072 with multiplicity 2, 0.5266 with multiplicity 2, 0.5519 with multiplicity 2, 0.5752 with multiplicity 2, and 0.5891 with multiplicity 2.

The *A*-, *D*- and *E*-efficiency measures of Λ_1 were obtained to be 0.6200, 0.6225, and 0.5691, respectively. For Λ_2, its *A*-, *D*- and *E*-efficiency measures were obtained to be 0.5799, 0.5816, and 0.5269, respectively. Λ_3 has its *A*-, *D*- and *E*-efficiency measures to be 0.6366, 0.5616, and 0.5130 respectively. Moreover, the *A*-, *D*- and *E*-efficiency measures of Λ_4 are 0.5483, 0.5492, and 0.5072, respectively.

Table 9 shows a summary of the c. e. f_s and *A*-, *D*- and *E*-efficiency measures for each design under consideration.

Table 9: Summary of the c. e. f_s and *A*-, *D*- and *E*-efficiency measures for each design

v	Design	c. e. f	Multiplicity of c. e. f	*A*-efficiency measure	*D*-efficiency measure	*E*-efficiency measure
5	Λ_1	0.5691 0.6809	2 2	0.6200	0.6225	0.5691
7	Λ_2	0.5269 0.5873 0.6358	2 2 2	0.5799	0.5816	0.5269
9	Λ_3	0.5130 0.5459 0.5833 0.6078	2 2 2 2	0.6366	0.5616	0.5130
11	Λ_4	0.5072 0.5266 0.5519 0.5752 0.5899	2 2 2 2 2	0.5483	0.5492	0.5072

DISCUSSION OF RESULTS

Notice from the vectors of eigenvalues presented in Section 4.3 that, for each scaled information matrix, L_i^*, where i = 1, . . . , 4, precisely one eigenvalue is zero, hence each design, Λ_i, i = 1, . . . , 4 is a connected design. From Table 9, for each RGSLR, Λ_i, i = 1, . . . , 4, there exists γ distinct c. e. f_s, where $\gamma \geq 2$ and each of the γ distinct c. e. f_s has the same multiplicity, 2. Moreover, the value of γ increases as the number of treatments (v), hence the size of the design, increases.

For the *A*-, *D*- and *E*-efficiency measures of each design, it is obvious from Table 9 that for Λ_i, i = 1, . . . , 4, the *E*-efficiency measure is the least amongst the three measures. However, for Λ_i, i = 1, 2, 4, the *D*-efficiency measure is higher than its corresponding *A*-efficiency measure, while the *A*-efficiency for Λ_3 is higher than its *D*-efficiency measure.

Moreover, the *D*- and *E*-efficiency measures of the designs decrease with a corresponding increase in the value of *v*, that is, as the design becomes larger. However, the *A*-efficiency measures of these designs do not seem to follow a fixed pattern as the size of the design increases (or decreases). In particular, the *A*-efficiency measure of Λ_3 is the highest, followed by that of Λ_1, then the one for Λ_2, while the least is from Λ_4.

CONCLUDING REMARKS

Each design under consideration, Λ_i, $i = 1, \ldots, 4$, is a connected design and has at least two distinct values of its c. e. f_s , where each distinct c. e. f has the same multiplicity, 2. Moreover, the number of distinct c. e. f_s increases as the design becomes bigger in size, that is, as the number of treatments increases. However, the *D*- and *E*-efficiency measures of these designs decrease as their sizes/number of treatments increase with no definite pattern for their *A*-efficiency measures.

The high values of the efficiency measures of these designs make them good designs for experiments in those experimental situations where they are suitable and designs of their sizes are required.

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