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An Extension Proof of Riemann Hypothesis by a Logical Entails Truth Table

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ABSTRACT: *There were many mathematicians who tried to prove or disprove the statement of Riemann Hypothesis. However, none of them have been successfully approved by the Clay Mathematical Institute. In addition, to the best of this author's knowledge, these mathematicians haven't employed the technique of logical truth table during their proofs. With reference to this author's previous proof in [1], this author have employed the method of multiplicative telescope together with the prime boundary gaps. In this extended version of my proof to the Riemann Hypothesis, this author tries to show that RH statement is true through the four cases of the conditional statements in the truth table. Three of the cases (I, II, IV) are found to be true for the conditional statement in the Riemann Hypothesis while only one (case III) is found to be false (and acts as the disproof by a counter-example). Moreover, there are also three sub-cases (i, ii, iii) [1] among these four tabled cases. The main idea is that the we may disproof the hypothesis statement that is similar to the RH one by first find a counter-example which is obviously a disproof (case III) to the (Riemann) hypothesis. But it is NOT compatible with the GÖdel's Incompleteness Theorem. Otherwise either the disproof to the statement or the Gödel is incorrect which is impossible. Hence, the disproof is said to be incompatible with the Gödel. On the other hand, all of the other truth cases (I, II, IV) for the statement are indeed the examples for the positive results to the Riemann Hypothesis statement and are compatible with the Gödel. Therefore, the only way to make a conclusion is to say or force the Riemann Hypothesis statement to be correct.In general, for any hypothesis with the conditional statements structure like the Riemann one, we may also prove them by the similar techniqe and the arguments of the truth table for their conditional statements together with the Gödel's Incompleteness theorem to force the positive result for the hypothesis statement. Actually, there are many applications for the truth tables especially in the fields like language (structure & modeling) or in engineering (logic gates & programming) etc during our everyday usage.*

KEYWORDS: extension, proof of Riemann hypothesis, logical entails truth table

INTRODUCTION

As shown in my previous papers [1], this author has developed a suggested way to prove the long awaiting question for the truthness of the Riemann Hypothesis in a long historic period of time. In fact, with reference to [1], it employs the method of multiplative telescope together with some logics.

Actually, one may extend [1] to prove the Riemann Hypothesis by the logical entails truth table in a novel and complete way. Then we may let the Riemann Hypothesis statement to be the hypothesis X and want to determine whether it may be correct or not. In fact, the truth table for the Riemann Hypothesis should like the following:

Hypothesis X	Consequent Y	Conditional X → Y
True (T)	True (T)	True (T)
False (F)	True (T)	True (T)
True (T)	False (F)	False (F)
False (F)	False (F)	True (T)

Figure 1: The Truth table for a logical conditional statement [2] & [3] for my proof in Riemann Hypothesis statement.

In reality, a truth table may have some applications in both language or engineering or one may refer to another story. Then, the Riemann Hypothesis statement is said to be both true (for the infinite many examples and compatible with GÖdel’s Incompleteness Theorem) or the positive proof and false (for the disproof by the counter-example and incompatible with the GÖdel’s Incompleteness Theorem). Thus, in such a case, the Riemann Hypothesis is forced to be true, otherwise either the disproof by the counter-example or the GÖdel’s Incompleteness Theorem may be false which cannot happen. All of the above results (true, false cases and the incompleteness) imply and force the Riemann Hypothesis statement must be true or correct. In practice, this author wants to remark that his University of Hong Kong’s undergraduate project was once about the discussion in searching the foundation of mathematics by the philosophy (i.e. mathematical logic-ism, intuitionism, formalism and the Gödel’s Incompleteness theorem etc where the aforementioned (intuitive) logical truth table (Figure 1) for the Riemann Hypothesis may have some relationship with the previous project’s research) from 1995 to 1996 before the Hong Kong handover. Indeed, my UG project’s topic was somehow implying a linking with the oil risk in 1970s and the associated commercial economic threats etc.

Literature Review -- A Proof for the infinite many of Riemann Non-Trivial Zeros
Proof for ininite number of Primes

We may show that there is infinite many of prime numbers with the algebraic philosophical proofs as outlined in [8] & [9]. Then we may apply the prime and non-trivial Zeros fourier duality [10] & [12] to prove that there are actually infinite many Riemann non-trivial zeta zeros. In practice, this author suggests one may find the alternative proof for infinite many number of primes by point set topology as outlined in [11], [12], [13], [14] & [15]. However, no matter what kind of proof(s) that you may prefer, the key focus is to find out the contradiction(s) behind the initial assumption(s) and the final result(s) that obtained etc.

Proof for Infinite number of non-trivial zeta zeros
The basic idea of the proof for infinite number of non-trivial zeta zeros by the analytic number theory is: we may first assume that there were finite number of non-trivial zeta zeros. Then from the Riemann Explicit formula [16]:

$\psi(x) = \sum_{p^k \le x} lnp = x - \sum_\rho \frac{x^\rho}{\rho} - ln(2\pi) - \frac{1}{2} ln(1 - x^{-2})$

where ρ is the complex Riemann Zeta zeros,

$\frac{d}{dx}\psi(x) = 1 - \sum_\rho x^{(\rho-1)} - \frac{(x^{-3})}{(1-x^{-2})} = 1 - \sum_\rho x^{(\rho-1)} - \frac{1}{x(x^2-1)}$

But as there were finite number of non-trivial zeta zeros, thus $\sum_\rho x^{(\rho-1)}$ = k where k is a complex valued constant, then ψ'(x) = 1 - (a+bi) - $\frac{1}{x(x^2-1)}$ where k' = (1-a) + bi

But the integration [17] of $\frac{1}{x(x^2-1)}$ = - ln |x| + ½ ln |x-1| + ½ ln |x+1| + c or

ψ (x) = - ln |x| + ½ ln |x-1| + ½ ln |x+1| + [(1-a)+bi]x + c

------ (computed ψ(x) formula)

$.\neq. \sum_{p^k \le x} lnp = \sum_{n \le x}$.Λ(n)

As the fact that

$\psi(x) = \sum_{p^k \le x} lnp = x - \sum_\rho \frac{x^\rho}{\rho} - ln(2\pi) - \frac{1}{2} ln(1 - x^{-2})$ -------- (original ψ(x) formula).

Actually, $\sum_\rho \frac{x^\rho}{\rho}$ is a polynomial with the highest degree of sup{ρ | ρ is a zeta zeros in the control strip}. But we are now only to employ a linear function of order one (i.e. [(1-a)+bi]x to approximate x - $\sum_\rho \frac{x^\rho}{\rho}$ or x - $\sum_\rho \frac{e^{\rho ln(x)}}{\rho}$. As $e^{\rho ln(x)}$ is an index-complex valued exponential function which may seem to make no sense if there were finite number of non-trivial zeros that may imply a linear approximation [(1-a)+bi]x to $e^{\rho ln(x)}$. Actually, the Taylor Expansion for $e^{\rho ln(x)}$ at point "a" for the first 3 terms, according to the U.S.A. Mathematic-a (Home Version, 2023) is:

$$e^{ln(a)^\rho} + e^{ln(a)^\rho}\rho ln(a)^{-1+\rho} ln'(a)(x-a) + \left(\frac{1}{2}e^{ln(a)^\rho}\rho ln(a)^{-2+\rho}\right)$$
$$\left(-ln'(a)^2 + \rho ln'(a)^2 + \rho ln(a)^\rho ln'(a)^2 + ln(a)ln''(a)\right)(x-a)^2 + \ldots + o(x-a)^4$$

which is a polynomial of order at least 3 for the summation to all of the complex valued zeta zeros.

So $\sum_\rho \frac{x^\rho}{\rho} = \sum_\rho \frac{e^{\rho ln(x)}}{\rho}$ where ρ = u + vi and belongs to those complex valued zeta zeros

$$=\{ \sum_{re^{i\theta}\in\rho}. [e^{ln(a)^\rho} + e^{ln(a)^\rho}\rho ln(a)^{-1+\rho} ln'(a)(x-a) + \left(\tfrac{1}{2}e^{ln(a)^\rho}\rho ln(a)^{-2+\rho}\right)$$
$$\left(-ln'(a)^2 + \rho ln'(a)^2 + \rho ln(a)^\rho ln'(a)^2 + ln(a)ln''(a)\right)(x-a)^2 + \ldots + o(x-a)^4$$
]/ρ}

In reality, the computed ψ(x) is also different from the original ψ(x) by a constant term ln(2π) = $\frac{\xi'(0)}{\xi(0)}$ where x = 0 or 2nπ for n = 1,2… but c in the computed ψ(x) formula may be equal to $\frac{\xi'(x)}{\xi(x)}$ where $0 < x. < 2n\pi$. Thus, a contradiction is occured mainly due to the initial assumption that there were just a finite number of non-trivial Riemann Zeta zeros. Hence, we may conclude that there are infinite number of Riemann non-trivial zeta zeros.

Alternatively, we have: $x(x^2-1)\psi'(x) = x(x^2-1) - (x^2-1)k - 1$

After simplification, $(x^2-1)[x(\psi'(x)-1)+k] = -1$

[x(ψ'(x)-1)+k] = -1 or [x(ψ'(x)-1)+k] = 1

x = 0 or x = $2^{1/2}$ or x = - ($2^{1/2}$) and ψ'(x) = $\frac{-k-1}{x}$ + 1 or ψ'(x) = $\frac{-k+1}{x}$ +1

ψ(x) = -(k+1) ln (x) + x + c or ψ(x) = -(k-1) ln (x) + x + c

$.\neq. \sum_{p^k \le x} lnp = \sum_{n \le x}$.Λ(n)

$= x - \sum_\rho \frac{x^\rho}{\rho} - ln(2\pi) - \frac{1}{2} ln(1 - x^{-2})$

which is a contradiction due to the initial assumption that there were finite number of non-trivial zeta zeros. Hence, we conclude that there are infinite number of non-trivial zeta zeros.

A Conversion between the primes and zeta zeros

In fact, we may still need to compute the recurrence formula for those primes by the Golomb's formula. Once if my proposed proof of Riemann Hypothesis is verified or found to be true, then we may compute the corresponding nth term of the Riemann Zeta zeros as shown in [18] & [19]. Theoretically, it is possible that we may compute the (mirror image inverse) n+1th term of the prime from the nth term of the Riemann Zeta zeros from a corresponding recurrence formula. However, the focus of my series of proof is to determine whether the Riemann Hypothesis is true or false. Thus, this author believes that these pair of the recurrence formulas for both primes and non-trivial zeta zeros are in fact out of the scope of the present research paper.

The Extended Proof to Riemann Hypothesis

Case I: the (assumption) truth for the Riemann Hypothesis statement gives a positive true result and hence implying RH is correct (i.e. true & true imply true – row one of Figure 1). The proof for the Riemann Hypothesis to be true has been shown as in my previous paper [4] by employing Matlab programming code for the verification all over the complex infinity plane except the line x = 1 which is a singularity and hence has an infinite many solutions;

Case II : the (assumption) false for the Riemann Hypothesis statement gives a positive true result and hence implying RH is correct (i.e. false & true imply true – row two of Figure 1). The proof for the Riemann Hypothesis to be true has been shown in my previous paper [1]'s case III or the sub-case iii. In reality, the zeta root model equation is true all over the critical strip up to infinity as in the process of computation [6], the assumption for the Riemann Zeta function (I.e. $\sum_{n=1}^{\infty}\frac{1}{n^s}$) is a summation from one to infinity, so as the calculated zeta root model equation should be validated all over the infinite real-complex plane (i.e. $\sum_{k=1}^{\infty}\frac{1}{e^{(u+vi)ln(k)}}-\frac{(u+vi)(x-k)}{ke^{(u+vi)ln(k)}}+\frac{\left(-\frac{u^2+2uvi+vi^2-u-vi}{2k^2}+\frac{(u+vi)^2}{k^2}\right)(x-k)^2}{e^{(u+vi)ln(k)}}$). In addition, we solve for the prescribed zeta root model equation to get the final zeta model equation answer 0.5 +/- $\{4*\cot[\ln(x)]/(x+1)^2\}$ for all x over the infinite real-complex plane together with infinite many solutions;

Case III: the (assumption) true for the Riemann Hypothesis statement gives a negative false result and hence implying RH is incorrect (i.e. true & false imply false – row three of Figure 1). The disproof will be shown as below:

First assume that $0.5+/-[4*\cot(\ln(x))/(x+1)^2]i$ is just the model equation for the Riemann Zeta function (i.e. $\sum_{n=1}^{\infty}\frac{1}{n^s}$). Then according to the Dirichlet-Eta function [5]:

$\eta(s)=\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s}$ where s = u+vi for all values on the complex plane, by applying the Taylor approximation to η(s) with Maple soft and solve it, we may get:

$$h := \frac{e^{(a-1)\pi I}}{e^{(u+vi)ln(a)}} + \frac{\left(e^{(a-1)\pi I}\pi I - \frac{e^{(a-1)\pi I}(u+vi)}{a}\right)(x-a)}{e^{(u+vi)ln(a)}} + \frac{\left(-\frac{e^{(a-1)\pi I}\pi^2}{2} - \frac{e^{(a-1)\pi I}(u^2+2uvi+vi^2-u-vi)}{2a^2} - \frac{e^{(a-1)\pi I}(\pi a I - u - vi)(u+vi)}{a^2}\right)(x-a)^2}{e^{(u+vi)ln(a)}}$$

where its roots are:

$$\left\{a = a, u = \frac{I\pi a^2 - I\pi ax - avi + vix - \frac{3a}{2} + \frac{x}{2} + \frac{\sqrt{-4I\pi a^3 + 8I\pi a^2 x - 4I\pi ax^2 + a^2 - 6xa + x^2}}{2}}{-x+a}, vi = vi, x = x\right\}, \left\{a = a, u = \frac{I\pi a^2 - I\pi ax - avi + vix - \frac{3a}{2} + \frac{x}{2} - \frac{\sqrt{-4I\pi a^3 + 8I\pi a^2 x - 4I\pi ax^2 + a^2 - 6xa + x^2}}{2}}{-x+a}, vi = vi, x = x\right\}$$

which is obviously different from the previous roots found in [6] for $\sum_{n=1}^{\infty}\frac{1}{n^s}$. Or

$$f := \frac{1}{e^{(u+vi)ln(k)}} - \frac{(u+vi)(x-k)}{ke^{(u+vi)ln(k)}} + \frac{\left(-\frac{u^2+2uvi+vi^2-u-vi}{2k^2} + \frac{(u+vi)^2}{k^2}\right)(x-k)^2}{e^{(u+vi)ln(k)}}$$

where its roots are:

$$\left\{k = k, u = \frac{-vik + vix - \frac{3k}{2} + \frac{x}{2} + \frac{\sqrt{k^2 - 6xk + x^2}}{2}}{-x+k}, vi = vi, x = x\right\}, \left\{k = k, u = \frac{-vik + vix - \frac{3k}{2} + \frac{x}{2} - \frac{\sqrt{k^2 - 6xk + x^2}}{2}}{-x+k}, vi = vi, x = x\right\}$$

In fact, both of the roots for $\sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s}$ are different. Actually, there is a nπ phase difference/shift between their roots as:

Roots for $\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s}$ is 0.5 +/- $\{4*\cot[n\pi - \ln(x)] / (x+1)^2\}$ but

roots for $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is 0.5 +/- $\{4*\cot[\ln(x)] / (x+1)^2\}$;

If we apply the Taylor approximation for cot(x) = (1/x) – (x/3) & ln(x) = 2(x-1)/(x+1) & takes the limit x tends to infinity, then we may have:

0.5 +/- {4*cot [nπ-ln(x)] / $(x+1)^2$} tends to
4*{[1/(nπ – 2)] – (1/3)(nπ-2)}/$(x+1)^2$
0.5 +/- {4*cot [ln(x)] / $(x+1)^2$} tends to (2/3)$[1/(x+1)]^2$
Obviously, $\sum_{n=1}^{\infty}\frac{-1^{(n-1)}}{n^s}$. ≠.0 due to the phase shift for s = 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i .
But $\sum_{n=1}^{\infty}\frac{1}{n^s}$ = 0 for s = 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i. This author have shown in [6] that s = 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i is the computed roots for $\sum_{n=1}^{\infty}\frac{1}{n^s}$. This result may thus imply that the Riemann

Hypothesis statement is incorrect which is the disproof to the RH by the aforementioned counter-example (just only one is enough but we may still have infinite many counter-examples as the Riemann Zeta root model equation always expands all over the real-complex plane with infinite many solutions;

Case IV: the (assumption) false for the Riemann Hypothesis statement gives a negative false result and hence implying RH is true (i.e. false & false imply true – row four of Figure 1). The proof has been shown in my previous paper [1]'s case I & II or the sub-case i & ii. In fact, the wrong assumption is just the number theory equations of (*') in [1] but NOT for the line x = 1 with infinite many solutions in the case of ξ(1) over the line x = 1. Indeed, the the prime gap (difference) answers in [1] are always with infinite many solutions NO matter what the initial assumption may be for all j lies between 1 and infinity over the line x = 1. Moreover, for the case {u + vi / 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i | u & v are real numbers}, it includes the set of all the infinite real-complex plane. Hence, the sub-case ii will have infinite many solutions as shown in the final prime gap (difference) answer in [1] NO matter what the initial assumption may be for all j lies between 1 and infinity within the infinite real-complex plane.

CONCLUSION

In a nutshell, this author have already proved the statement of Rienamm Hypothesis is true mainly by the multiplicative telescopic method together with the differences in prime gaps [1] and the logical truth table (Figure 1). As shown in the previous proof section, there are infinite many cases or examples (I, II, IV) for the Riemann Hypothesis is said to be true but we still cannot conclude that RH is true because of the GÖdel's Incompleteness Theorem. Thus, we may just say that these truth examples (case I, II, IV) of RH are only compatible true examples [27] with the GÖdel's Incompleteness Theorem. However, there is only one case (III) for the Riemann Hypothesis to be false or RH is disproved by the counter-example [26] in case III. Then in such a case, RH is said to be false or it is the disproof of the Riemann Hypthesis (by a counter-example) which is NOT compatible with the Gödel's undecidability. Otherwise, either my disproof to RH in case III or the GÖdel's Incompleteness Theorem will be false which are both impossible [28]. Therefore, only the infinite many true examples of Riemann Hypothesis must be correct and compatible with the Gödel's undecidability. Or in such a case, the Riemann Hypothesis is thus forced to be correct [29 a & b]. Therefore, we come to a conclusion that this author have proved that the Riemann Hypothesis statement is correct or true.

A last word for this author's final remark is that the Riemann Hypothesis statement may be independent of the present ZFC system as one may need to reconstruct a new real number line where x = 0.5 becomes x = 0 in order to contain all non-trivial zeta zeros. Once if we may restore the traditional real number line without a transformation of x = 0.5 to x = 0, then the Riemann Hypothesis statement may depend on the old ZFC system. Hence, we may need to employ a (hybrid) fuzzy between the independence and dependence or a "(Hybrid) Fuzzy ZFC system" [30] for solving the Riemann Hypothesis. In practice, no matter the dependence or the independence, the key interests will be to further investigate both the structure and the random-ness of those non-trivial zeta zeros together with an application in the (quantum) cryptography etc.

Limitations -- An Error Estimate for the Computed Root's Model Equation

In practice, the computed Riemann Zeta function (i.e. $\sum_{n=1}^{\infty}\frac{1}{n^s}$)'s root model equation is 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i. It tends to (2/3)$[1/(x+1)]^2$ when x tends to infinity for cot(ln(x)) where cot(x) = (1/x) – (x/3) & ln(x) = 2(x-1)/(x+1).

Actual Zeta Root	Estimated Zeta Root	Absolute Percentage Error
14.1347	14 -- (for (2/3)$[1/(x+1)]^2$ = [14.1347])	0.95297%
21.0220	21 -- (for (2/3)$[1/(x+1)]^2$ = [21.0220])	1.04%
25.0109	25 -- (for (2/3)$[1/(x+1)]^2$ = [25.0109])	0.0436%
30.4249	30 (for (2/3)$[1/(x+1)]^2$ = [30.429])	1.40984%
32.9351	33 (for (2/3)$[1/(x+1)]^2$ = [32.9351])	0.1971%
37.5862	38 (for (2/3)$[1/(x+1)]^2$ = [37.5862])	1.1009%
40.9187	41 (for (2/3)$[1/(x+1)]^2$ = [40.9187])	0.1987%
43.3271	43 (for (2/3)$[1/(x+1)]^2$ = [43.3271])	0.75495%
48.0052	48 (for (2/3)$[1/(x+1)]^2$ = [48.0052])	0.01083%
49.7738	50 (for (2/3)$[1/(x+1)]^2$ = [49.7738])	0.45446%

Figure 2: Absolute Percentage Error between the first to the tenth estimated (by s = 0.5+/-[4*cot(ln(x)) / $(x+1)^2$]i) & the actual Riemann zeta root.

It seems that the maximum absolute percentage error is about 1.5% for the first ten Riemann Zeta zeros which lies in an acceptable range. This also constitues another positive result for the disproof for Riemann Hypothesis by the counter-example. In fact details for the structure of the Riemann Zeta zeros should be investigated by the complex Lie Algebra and Lie Groups [21] together with the corresponding Branching rule(s) [20] & [23] discovered by the mathematic-a software LieART 2.0 [22] or even the complex Lie Algebra as well as the homology & homotopy etc [24]. In fact, by considering the lattice's weight unification and decomposition so as to understand the complexity of the processing system [25]. Actually, the determination of the structure for the Riemann Zeta zeros is another story for researching which is out of the scope of the present topic in the dtermination whether the statement of Riemann Hypothesis is true or false.

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