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Initial Value Solvers for Direct Solution of Fourth Order Ordinary Differential Equations in a Block from Using Chebyshev Polynomial as Basis Function

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ABSTRACT: *The numerical computation of fourth order ordinary differential equations cannot be gloss over easily due to its significant and importance. There have been glowing needs to find an appropriate numerical method that will handle effectively fourth order ordinary differential equations without resolving such an equation to a system of first order ordinary differential equations. To this end, this presentation focuses on direct numerical computation to fourth order ordinary differential equations without resolving such equations to a system of first order ordinary differential equations. The method is not predictor – corrector one due to its limitation in the level of accuracy. The method is order wise christened "Block Method" which is a self-starting method. In order to achieve this objective, Chebyshev polynomial is hereby used as basis function.*

KEY WORDS: Predictor, Corrector, Chebyshev, Consistency, zero – stability, multistep, continuous.

INTRODUCTION

The quest for numerical solution to differential equations cannot be over emphasized due to their desirability because the analytical solutions to some of these differential equations are intractable. In order to circumvent this problem hence come the approximate solutions. Many authors have proposed several methods such as single step methods which include among others the Euler's

Method, Heuns Method, Picard iterative method and Classical Runge – Kutta method to mention just a few.

In this paper, an attempt shall be made to derive a Block linear multi – step method for solving fourth order initial value problems of the form

$$y'''' = f(x, y, y', y'', y'''), \quad (1)$$

together with the following conditions

$$y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma, y'''(a) = \delta \quad (2)$$

without resolving such an equation to a system of first order ordinary differential equations.

So many authors have proposed linear multi - step method to solve second order ordinary differential equations directly in which most of them are predictor – corrector in nature. This is an improvement over and above the known single step methods. See [1, 2, 3, 4, 5, 6, 9, and 10]. Howbeit the methods has better accuracy over the single step method but it equally suffers some set back such as it depend on single step methods to determine its starting value apart from being too laborious to develop an appropriate corrector method .

In order to circumvent the hurdle being faced in developing predictor – corrector method, researchers came up with a noble method known as Block Linear Multi- step method. This method has an advantage over the predictor – corrector method in which the method is self starting, that is it does not depend on additional information in getting its starting value. Some of the authors that have worked extensively on this method include among others. See [7, 10, 13, 14, 15, and 16] Since the Block method is a self starting method; this increases the level of accuracy of result over and above the predictor – corrector methods which depend on another method to get its starting value.

In deriving the Block method as initial value solvers, some authors only gave the discrete formula which did not allow evaluation at all points within the interval of integration otherwise known as the interior points. It only allows evaluation at the grid points. In this presentation, an attempt is made to present both the continuous and its equivalent of the discrete formulae. The continuous formula allows evaluation at all points within the interval of integration. Another noble feature of the method under consideration is that it allows the integration of differential equations of order four to be carried out directly without reducing the equation to a system of first order ordinary differential equations before evaluating the integration.

DERIVATION OF THE METHOD

A continuous Block multistep method for solving fourth order ordinary differential equations is hereby presented. The method will be generated using Chebyshev polynomials as basis function. The choice of Chebyshev polynomial above all other monomials is as a result of its level of accuracy among other monomials. See [8]

Chebyshev polynomial according to Fox and Parker [12] is defined as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1(1)n \quad (3)$$

Where

$$T_0(x) = 1 \ and \ T_1(x) = 2 \quad (4)$$

At this point, a four step multistep method is proposed such that $x_n \leq x \leq x_{n+4}$

Let the analytic solution to differential equation

$$y'''' = f(x, y, y', y'', y'''), y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma, y'''(a) = \delta \quad (5)$$

be approximated by the polynomial

$$y_{(x)} = \sum_{n=0}^{k} a_r T_r(x) \quad (6)$$

On a partition $a = x_0 < x_1 < x_2 < \ . \ . \ . < x_k = b$ with a constant step size and $T_r(x)$ can be recursively generated by the use of equation (1) and noting that

$$T_0(x) = 1 \ and \ T_1(x) = x \ . \quad (7)$$

At this point, consider the linear multistep method

$$\sum_{j=0}^{3} \alpha_j y_{n+j} = h^4 \sum_{j=0}^{4} \beta_j f_{n+j} \quad (8)$$

where $\alpha_j \ and \ \beta_j$ are constants. By the virtue of equation (3) with $k = 9$, this yields

$$y(x) = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4(8x^4 - 8x^2 + 1) + a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 48x^4 + 18x^2 - 1) + a_7(64x^7 - 112x^5 + 56x^3 -$$

$7x) + a_8(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1)$

(9) With the transformation $x = \frac{2x-b-a}{b-a}$ where $a < x < b$ and evaluate equation (9) in the closed interval $x_n \le x \le x_{n+4}$ yields

$$y(x) = a_0 + a_1\left(\frac{x-kh-2h}{h}\right) + a_2\left\{2\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} + a_3\left\{4\left(\frac{x-kh-2h}{2h}\right)^3 - 3\left(\frac{x-kh-2h}{2h}\right)\right\} + a_4\left\{8\left(\frac{x-kh-2h}{2h}\right)^4 - 8\left(\frac{x-kh-2h}{2h}\right)^2 + 1\right\} + a_5\left\{16\left(\frac{x-kh-2h}{2h}\right)^4 - 20\left(\frac{x-kh-2h}{2h}\right)^2 + \left(\frac{x-kh-2h}{2h}\right)\right\} + a_6\left\{32\left(\frac{x-kh-2h}{2h}\right)^6 - 48\left(\frac{x-kh-2h}{2h}\right)^4 + 18\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} + a_7\left\{64\left(\frac{x-kh-2h}{2h}\right)^7 - 112\left(\frac{x-kh-2h}{2h}\right)^5 + 56\left(\frac{x-kh-2h}{2h}\right)^3 - 7\left(\frac{x-kh-2h}{2h}\right)\right\} + a_8\left\{128\left(\frac{x\,kh-2h}{2h}\right)^8 - 256\left(\frac{x\,kh-2h}{2h}\right)^6 + 160\left(\frac{x\,kh-2h}{2h}\right)^4 - 32\left(\frac{x\,kh-2h}{2h}\right)^2 + 1\right\} \quad (10)$$

The fourth derivative of equation (10) gives

$$y''''(x) = \frac{12a_4}{h^4} + \frac{120a_5}{h^4}\left(\frac{x-kh-2h}{2h}\right) + \frac{72a_6}{h^4}\left\{10\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} + \frac{840}{h^4}\left\{4\left(\frac{x-kh-2h}{2h}\right)^3 - \frac{x-kh-2h}{2h}\right\} + \frac{240}{h^4}\left\{\frac{56}{h^4}\left(\frac{x-kh-2h}{2h}\right)\right\} \quad (11)$$

Interpolating equation (10) at $x = x_{n+k}$, $k = 0\ (1)3$ and collocate equation (11) at

$x = x_{n+k,}\ k = 0\ (\ 1)4$ yields the system of linear equations

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 \\ & & & & 12 & -60 & 108 & 0 & -360 \\ & & & & 12 & -120 & 468 & -2520 & 7920 \\ & & & & 12 & 0 & -720 & 0 & 2400 \\ & & & & 12 & 60 & 108 & 0 & -360 \\ & & & & 12 & 120 & 468 & 2520 & 7920 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad (12)$$

Solving equation (12) using Maple software yields

$$a_0 = \frac{1}{2520}(2520y_{n+1} - 2520y_{n+2} + 2520y_{n+3} - h^4 f_n + 67h^4 f_{n+1} + 288h^4 f_{n+2} + 67h^4 f_{n+3} - h^4 f_{n+4})$$

$$a_1 = \frac{1}{7560}(-5040y_n + 7560y_{n+1} - 15120y_{n+2} + 12600y_{n+3} - 2h^4 f_n + 494h^4 f_{n+1} + 1659h^4 f_{n+2} + 374h^4 f_{n+3} - 5h^4 f_{n+4})$$

$$a_2 = \frac{1}{60480}(60480y_{n+1} - 120960y_{n+2} + 60480y_{n+} - 31h^4 f_n + 2476h^4 f_{n+1} + 10230h^4 f_{n+2} + 2476h^4 f_{n+3} - 31h^4 f_{n+4})$$

$$a_3 = \frac{1}{4230}(-1440y_n + 4320y_{n+1} - 4320y_{n+2} + 1440y_{n+3} - h^4 f_n + 124h^4 f_{n+1} + 474h^4 f_{n+2} + 124h^4 f_{n+3} - 31h^4 f_{n+4})$$

$$a_4 = \frac{h^4}{60}(f_{n+1} + 3f_{n+2} + f_{n+3})$$

$$a_5 = \frac{h^4}{120}(-f_{n+1} + f_{n+3})$$

$$a_6 = \frac{h^4}{30240}(5f_n + 64f_{n+1} - 138f_{n+2} + 64f_{n+3} + 5f_{n+4})$$

$$a_7 = \frac{h^4}{5040}(-f_n + 2f_{n+1} - 2f_{n+3} + f_{n+4})$$

$$a_8 = \frac{h^4}{20160}(f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4})$$

Using the value of a's in equation (10) yields the continuous scheme taken into consideration that

$$t = \frac{x-kh-2h}{2h}$$

$$y(x) = \frac{y_n}{60480}(-80640t^3 - 20160t) + \frac{y_{n+1}}{60480}(241920t^3 + 120960t^2 - 120960t) + \frac{y_{n+2}}{60480}(-241920t^3 - 241920t^2 + 60480t + 60480) + \frac{y_{n+3}}{60480}(80640t^3 + 120960t^2 + 40320t) + \frac{h^4 f_n}{60480}(384t^8 - 768t^7 - 448t^6 + 1344t^5 - 728t^3 + 22t^2 + 110t) + \frac{h^4 f_{n+1}}{60480}(-1536t^8 + 1536t^7 + 7168t^6 - 10752t^5 + 18368t^3 - 423t^2 - 3944t) + \frac{h^4 f_{n+2}}{60480}(2304t^8 - 13440t^7 + 40320t^4 + 26544t^3 - 9276t^2 - 6636t) + \frac{h^4 f_{n+3}}{60480}(-1536t^8 - 1536t^7 + 7168t^6 + 10752t^5 - 4480t^3 - 424t^2 + 472t) + \frac{h^4 f_{n+4}}{60480}(384t^8 + 768t^7 - 448t^6 - 1344t^5 + 616t^3 + 22t^2 - 82t) \quad (13)$$

From the linear multistep method

$$y(x) = \sum_{j=0}^{3} \alpha_j y_{n+j} + h^4 \sum_{j=0}^{4} \beta_j f_{n+j} \quad (14)$$

Further simplification gives the following values of α_j and β_j

$$\alpha_0(t) = -\frac{4}{3}t^3 + \frac{t}{3}$$

$$\alpha_1(t) = 4t^3 + 2t^2 - 2t$$

$$\alpha_2(t) = -4t^3 - 4t^2 + t + 1$$

$$\alpha_3(t) = \frac{4}{3}t^3 + 2t^2 + \frac{2}{3}t$$

$$\beta_0(t) = \frac{1}{30240}(192t^8 - 384t^7 - 224t^6 + 672t^5 - 364t^3 + 11t^2 + 55t)$$

$$\beta_1(t) = \frac{1}{60480}(-1536t^8 + 1536t^7 + 7168t^6 - 10752t^5 + 18368t^3 - 423t^2 - 3944t)$$

$$\beta_2(t) = \frac{1}{60480}(2304t^8 - 13440t^6 + 40320t^4 + 26544t^3 - 9276t^2 - 6636t)$$

$$\beta_3(t) = \frac{1}{60480}(-1536t^8 - 1536t^7 + 7168t^6 + 10752t^5 - 4480t^3 - 424t^2 + 472t)$$

$$\beta_4(t) = \frac{1}{60480}(384t^8 + 768t^7 - 448t^6 - 1344t^5 + 616t^3 + 22t^2 - 82t)$$

Evaluating equation (13) at $x = x_{n+4}$, bearing in mind that $t = \frac{x-kh-2h}{2h}$ yield the discrete scheme which is the main method.

$$y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + \frac{h^4}{720}(f_{n+4} - 124f_{n+3} - 474f_{n+2} - 124f_{n+1} + f_n) \quad (15)$$

Equation (13) is the continuous scheme while equation (15) is the discrete scheme. The first derivative of the continuous function yields

$$\alpha'_0(t) = \frac{-2}{h}t^2 + \frac{1}{6h}$$

$$\alpha'_1(t) = \frac{6}{h}t^2 + \frac{2}{h}t - \frac{1}{h}$$

$$\alpha'_2(t) = -\frac{6}{h}t^2 - \frac{4}{h}t + \frac{1}{2h}$$

$$\alpha'_3(t) = \frac{2}{h}t^2 + \frac{2}{h}t + \frac{1}{3h}$$

$$\beta'_0(t) = \frac{1}{30240}\left(\frac{768}{h}t^7 - \frac{1344}{h}t^6 - \frac{672}{h}t^5 + \frac{1680}{h}t^4 - \frac{546}{h}t^2 + \frac{11}{h}t + \frac{55}{2h}\right)$$

$$\beta'_1(t) = \frac{1}{60480}\left(-\frac{6144}{h}t^7 + \frac{5376}{h}t^6 + \frac{21504}{h}t^5 - \frac{26880}{h}t^4 + \frac{27552}{h}t^2 - \frac{423}{h}t - \frac{1972}{h}\right)$$

$$\beta'_2(t) = \frac{1}{60480}\left(\frac{9216}{h}t^7 - \frac{40320}{h}t^5 + \frac{80640}{h}t^3 + \frac{39816}{h}t^2 - \frac{9276}{h}t - \frac{3318}{h}\right)$$

$$\beta'_3(t) = \frac{1}{60480}\left(-\frac{6144}{h}t^7 - \frac{5376}{h}t^6 + \frac{21504}{h}t^5 + \frac{26880}{h}t^4 - \frac{6720}{h}t^2 - \frac{423}{h}t + \frac{236}{h}\right)$$

$$\beta'_4(t) = \frac{1}{60480}\left(\frac{1536}{h}t^7 + \frac{2688}{h}t^6 - \frac{1344}{h}t^5 - \frac{3360}{h}t^4 + \frac{924}{h}t^2 + \frac{22}{h}t - \frac{41}{h}\right)$$

The second derivative of the continuous scheme gives

$$\alpha''_0(t) = -\frac{2}{h^2}t$$

$$\alpha''_1(t) = \frac{6}{h^2}t + \frac{1}{h^2}$$

$$\alpha''_2(t) = -\frac{6}{h^2}t - \frac{2}{h^2}$$

$$\alpha''_3(t) = \frac{2}{h^2}t + \frac{1}{h^2}$$

$$\beta''_0(t) = \frac{1}{30240h}\left(\frac{2688}{h}t^6 - \frac{4032}{h}t^5 - \frac{1680}{h}t^4 + \frac{3360}{h}t^3 - \frac{546}{h}t + \frac{11}{2h}\right)$$

$$\beta''_1(t) = \frac{1}{60480h}\left(-\frac{22504}{h}t^6 + \frac{16128}{h}t^5 + \frac{53760}{h}t^4 - \frac{53760}{h}t^3 + \frac{27552}{h}t - \frac{423}{2h}\right)$$

$$\beta''_2(t) = \frac{1}{60480h}\left(\frac{32256}{h}t^6 - \frac{100800}{h}t^4 + \frac{120960}{h}t^2 + \frac{39816}{h}t - \frac{4638}{h}\right)$$

$$\beta''_3(t) = \frac{1}{60480h}\left(-\frac{21504}{h}t^6 - \frac{16128}{h}t^5 + \frac{53760}{h}t^4 + \frac{53760}{h}t^3 - \frac{6720}{h}t - \frac{423}{2h}\right)$$

$$\beta''_4(t) = \frac{1}{60480h}\left(\frac{5376}{h}t^6 + \frac{8064}{h}t^5 - \frac{3360}{h}t^4 - \frac{6720}{h}t^3 + \frac{924}{h}t + \frac{11}{h}\right)$$

The third derivative of equation (9) yields

$$\alpha'''_0(t) = -\frac{1}{h^3}$$

$$\alpha'''_1(t) = \frac{3}{h^3}$$

$$\alpha'''_2(t) = -\frac{3}{h^3}$$

$$\alpha'''_3(t) = \frac{1}{h^3}$$

$$\beta'''_0\,(t) = \frac{1}{32040h^2}\left(\frac{8064}{h}t^5 - \frac{10080}{h}t^4 - \frac{3360}{h}t^3 + \frac{5040}{h}t^2 - \frac{273}{h}\right)$$

$$\beta'''_1\,(t) = \frac{1}{60480h^2}\left(-\frac{64512}{h}t^5 + \frac{40320}{h}t^4 + \frac{107520}{h}t^3 - \frac{80640}{h}t^2 + \frac{13776}{h}\right)$$

$$\beta'''_2\,(t) = \frac{1}{60480h^2}\left(\frac{96768}{h}t^5 - \frac{201600}{h}t^3 + \frac{120960}{h}t + \frac{19908}{h}\right)$$

$$\beta'''_3\,(t) = \frac{1}{60480h^2}\left(-\frac{64512}{h}t^5 - \frac{40320}{h}t^4 + \frac{107520}{h}t^3 + \frac{80640}{h}t^2 - \frac{3360}{h}\right)$$

$$\beta'''_4\,(t) = \frac{1}{60480h^2}\left(\frac{16128}{h}t^5 + \frac{20160}{h}t^4 - \frac{6720}{h}t^3 - \frac{10080}{h}t^2 + \frac{462}{h}\right)$$

Evaluating the first derivative at $x = x_{n+k},\ \ k = 0\,(1)4$ yields the following schemes respectively

$$hy'_n + \frac{11}{6}y_n - 3y_{n+1} + \frac{3}{2}y_{n+2} - \frac{1}{3}y_{n+3} = \frac{h^3}{60480}(-579f_n - 10861f_{n+1} - 3762f_{n+2} + 83f_{n+3} - 2691f_{n+4})$$

$$hy'_{n+1} + \frac{1}{3}y_n + \frac{1}{2}y_{n+1} - y_{n+2} + \frac{1}{6}y_{n+3} = \frac{h^3}{60480}\left(-31f_n + \frac{12535}{2}f_{n+1} + 2382f_{n+2} - \frac{521}{2}f_{n+3} + 41f_{n+4}\right)$$

$$hy'_{n+2} - \frac{1}{6}y_n + y_{n+1} - \frac{1}{2}y_{n+2} - \frac{1}{3}y_{n+3} = \frac{h^3}{60480}(55f_n - 1972f_{n+1} - 3318f_{n+2} + 236f_{n+3} - 41f_{n+4})$$

$$hy'_{n+3} + \frac{1}{3}y_n - \frac{3}{2}y_{n+1} + 3y_{n+2} - \frac{11}{6}y_{n+3} = \frac{h^3}{60480}\left(-14f_n + \frac{7465}{2}f_{n+1} + 10890f_{n+2} + \frac{1309}{2}f_{n+3} + 3f_{n+4}\right)$$

$$hy'_{n+4} + \frac{11}{6}y_n - 7y_{n+1} + \frac{19}{2}y_{n+2} - \frac{13}{3}y_{n+3} = \frac{h^3}{60480}(-151f_n + 19013f_{n+1} + 76758f_{n+2} + 29957f_{n+3} + 425f_{n+4})$$

Evaluation of second derivative at the following discrete points $x = x_{n+k},\ k = 0\ (1)4$ leads to the following additional schemes respectively

$$h^2y''_n - 2y_n + 5y_{n+1} - 4y_{n+2} + y_{n+3} = \frac{h^2}{60480}\left(4463f_n + \frac{84249}{2}f_{n+1} + 7962f_{n+2} + \frac{2265}{2}f_{n+3} - 241f_{n+4}\right)$$

$$h^2y''_{n+1} - y_n + 2y_{n+1} - y_{n+2} = \frac{h^2}{60480}\left(-157f_n - \frac{9495}{2}f_{n+1} - 102f_{n+2} - 87f_{n+3} - 11f_{n+4}\right)$$

$$h^2y''_{n+2} - y_{n+1} + 2\,y_{n+2} - y_{n+3} = \frac{h^2}{60480}(11f_n - 423f_{n+1} - 4638f_{n+2} - 423f\ n+_3 + 11f\ n+_4)$$

$$h^2y''_{n+3} + y_n - 4y_{n+1} + 5\,y_{n+2} - 2y_{n+3} = \frac{h^2}{60480}\left(-73f_n + \frac{20745}{2}f_{n+1} + 39714f_{n+2} + \frac{11337}{2}f_{n+3} - 241f_{n+4}\right)$$

$$h^2y''_{n+4} + 2y_n - 7y_{n+1} + 8\,y_{n+2} - 3y_{n+3} = \frac{h^2}{60480}\left(-409f_n + \frac{43929}{2}f_{n+1} + 87594f_{n+2} + \frac{125913}{2}f_{n+3} + 4295f_{n+4}\right)$$

The valuation of the third derivative at the following grid points $x = x_{n+k},\ k = 0\ (1)4$ yields the following additional schemes respectively.

$$h^3y'''_n + y_n - 3y_{n+1} + 3\,y_{n+2} - y_{n+3} = \frac{h}{60480}(-20034f_n - 97104f_{n+1} + 3780f_{n+2} - 6048f_{n+3} + 1134f_{n+4})$$

$$h^3y'''_{n+1} + y_n - 3y_{n+1} + 3\,y_{n+2} - \,y_{n+3} = \frac{h}{60480}(1050f_n - \,35952f_{n+1} - \,18396f_{n+2} + \,2856f_{n+3} - 462f_{n+4})$$

$$h^3y'''_{n+2} + y_n - 3y_{n+1} + 3\,y_{n+2} - \,y_{n+3} = \frac{h}{60480}(-546f_n + \,13776f_{n+1} + \,19908f_{n+2} - \,3360f_{n+3} + 462f_{n+4})$$

$$h^3y'''_{n+3} + y_n - 3y_{n+1} + 3\,y_{n+2} - \,y_{n+3} = \frac{h}{60480}(378f_n + \,672f_{n+1} + \,58212f_{n+2} + \,25704f_{n+3} - 1134f_{n+4})$$

$$h^3y'''_{n+4} + y_n - 3y_{n+1} + 3\,y_{n+2} - \,y_{n+3} = \frac{h}{60480}(-1218f_n + \,16464f_{n+1} + \,36036f_{n+2} + \,79968f_{n+3} + 19950f_{n+4})$$

In writing the derived method using Shampine and Watts (1969) block formula defined as

$$Ay_n = hBF(y_n) + \,ey_n + \,hdf_n$$

Where

$$A = (a_{ij}),\; B = (b_{ij}),\; e = (e_1, e_2\; .\;\;.\;\;.\; e_n)^T, d = (d_1,\; d_2\;\;.\;\;.\;\;.\; d_n)^T$$

$$y_n = (y_{n+1},\;\; y_{n+2},\;\;.\;\;.\;\;.\; y_{n+r})^T,\; and\; F(y_n) = (f_{n+1},\; f_{n+2},\;\;.\;\;.\;\;.\; f_{n+r})^T$$

Accordingly the value of , $A, B, d\; and\; e$ are as express below

A =

$$
\begin{pmatrix}
-4 & \overset{6}{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{2} & -3 & \frac{11}{6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-1}{2} & 1 & \frac{-1}{6} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \frac{-3}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{19}{2} & -7 & \frac{11}{6} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 5 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -7 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

D =
$\frac{1}{60480}(84 - 579 \ - 31 \ 55 \ - 14 \ - 151 \ 4463 \ - 157 \ 11 \ - 73 \ - 409 \ - 20034 \ 1050 \ - 546 \ 378 \ -$

$$E = \begin{pmatrix} 4 & 0 & 0 & 0 \\ \frac{-11}{6} & -1 & 0 & 0 \\ \frac{-1}{3} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{-11}{6} & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$-1 \quad 0 \quad 0 \quad 0$

$$B = \frac{1}{120960}\begin{pmatrix} -20832 & -79632 & -20832 & 168 \\ 21722 & -7524 & 166 & 5382 \\ 12535 & 4764 & -521 & 82 \\ 3944 & -6636 & 472 & -82 \\ 7465 & 21780 & 1309 & 6 \\ 38026 & 153516 & 59914 & 850 \\ 84249 & 15924 & 4530 & 482 \\ -9495 & -204 & -174 & -22 \\ -846 & -9276 & -846 & 22 \\ 20745 & 79428 & 11337 & -482 \\ 43929 & 175188 & 125913 & 8590 \\ -194208 & 7560 & -12096 & 2268 \\ -71904 & -36792 & -12096 & -924 \\ 27552 & 39816 & -6720 & 924 \\ 1344 & 116424 & 51408 & -2268 \\ 32928 & 72072 & 159936 & 39900 \end{pmatrix}$$

3 ANALYSIS OF METHOD

At this juncture, in investigation of some basic properties of the linear multistep method

$$\sum_{j=0}^{3} \alpha_k y_{n+k=} \; h^4 \sum_{j=0}^{4} \beta_k f_{n+k}$$

In particular the multistep method

$$y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + \frac{h^4}{720}(f_{n+4} - 124f_{n+3} - 474f_{n+2} - 124f_{n+1} + f_n) \quad (15)$$

in terms of order, convergence and error constant of the method. In order to investigate the order of the method, according to Lambert, [17] the linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}$$

associated with the linear difference operator

$$\mathcal{L}[y(x);h] = \sum_{j=0}^{k} \left[\alpha_j y\,(x+jh) - h^2 \beta_j y''\,(x+jh)\right]$$

Where $y(x)$ is an arbitrary function, continuously differentiable on an interval $[a,b]$. On assumption that $y(x)$ has as many higher derivatives as required, then on Taylor expansion about the point x , this expression was achieved

$$\mathcal{L}[y(x);h] = C_0 y(x) + C_1 h y^{(1)}(x) + \ .\ .\ . + C_q h^q y^{(q)}(x) + \ .\ .\ .$$

Where

$$C_0 = \sum_{j=0}^{k} \alpha_k, \quad C_1 = \sum_{j=1}^{k} j\alpha_k$$

$$C_2 = \frac{1}{2\,!} \sum_{j=0}^{k} j^k \alpha_k - \sum_{j=1}^{k} \beta_k$$

$$C_q = \frac{1}{q\,!} \sum_{j=0}^{k} j^q \alpha_k - \frac{1}{(q-2)\,!} \sum_{j=1}^{k} j^{q-2} \beta_k\,, \quad q = 3(1)n$$

Then the method is of order p if

$$C_0 = C_1 = \ .\ .\ .\ C_p = C_{p+1} = 0 \; and \; C_{p+2} \neq 0$$

Then C_{p+2} is the error constant while $C_{p+2} h^{p+2} y^{(p+2)}(x)$ is the principal local truncation error. Intuitively, applying the above principle to the linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^4 \sum_{j=0}^{k} \beta_j f_{n+j}$$

associated with the linear difference operator

$$\mathcal{L}[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j y\,(x + jh) - h^4 \beta_j y^{iv}\,(x + jh)\right]$$

Then

$$C_0 = \sum_{j=0}^{k} \alpha_k, \quad C_1 = \sum_{j=0}^{k} j\alpha_k\,, C_2 = \sum_{j=0}^{k} j^2\alpha_k\,, C_3 = \sum_{j=0}^{k} j^3\alpha_{k,}\, C_4 = \frac{1}{4\,!}\sum_{j=1}^{k} j^4\alpha_k - \sum_{j=1}^{k} \beta_k$$

$$C_q = \frac{1}{q\,!}\sum_{j=0}^{k} j^q\alpha_k - \frac{1}{(q-4)\,!}\sum_{j=1}^{k} j^{q-4}\beta_k\,, \;\; q = 5(1)n$$

Then the method is of order p if

$$C_0 = C_1 = \;.\;.\;.\; C_p = C_{p+1} = 0 = C_{p+3} = 0 \;\; but \; C_{p+4} \neq 0$$

Then C_{p+4} is the error constant while $C_{p+4}h^{p+4}y^{(p+4)}(x)$ is the principal local truncation error. On investigating the order of the scheme, it was found that the scheme is of order 6 with error constant of $C_{p+4} = \frac{1}{3024}$.

Theorem: The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero stable.

Consistency of a linear multistep method is when the method has order greater or equal to one, while zero stability deals with root of the first characteristics polynomial in which that none of its roots must has modulus greater that one and every root with modulus one is simple.[17]

At this juncture, the roots of the first characteristics polynomial is calculated thus

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$$

The equation has root of

$(r-1)^4 = 0$, $r = 1$ four times

This shows that the method under consideration is both consistent and zero stable, hence the method has ability to converge.

4 NUMERICAL COMPUTATION:

At this juncture, some numerical computations shall be carried out in order to demonstrate the convergence of the method.

Problem 1: Solve the differential equation

$$y'''' - 3y''' + 6y'' - 12y' + 8y.$$

Subject to the following conditions

$$y(0) = 1, y'(0) = 2, y''(0) = 0 \; and \; y'''(0) = 4$$

The analytic solution is

$$y\,(x) = 1.5e^{x} - 2.4e^{2x} + 0.9\cos 2x + 0.7\sin 2x$$

Problem 2 : Solve the initial value problem

$$2y'''' - 3y''' - 3y'' - 20y' + 12y$$

Subject to

$$y(0) = 1, y'(0) = -2\,, y''^{(0)} = 3 \; and \; y'''(0) = 4$$

The theoretical solution is

$$y(x) = \frac{1}{4}e^{2x} + \frac{43}{20}e^{-2x} - \frac{5}{7}e^{-3x} - \frac{24}{35}e^{\frac{x}{2}}$$

Problem 3: Compute the solution to the differential equation:

$$6y'''' - y''' - 23y'' - 4y' + 12y$$

Subject to

$$y(0) = -1, y'(0) = 2, y''(0) = -3, y'''(0) = -2$$

The analytical solution is

$$y(x) = \frac{1}{15}e^{-x} - \frac{120}{91}e^{-\frac{3}{2}x} - \frac{5}{84}e^{2x} + \frac{81}{260}e^{\frac{2}{3}x}$$

Problem 4: Solve the differential equation:

$$y'''' + 3y'' + (2 + \varepsilon\cos\omega t)y : t \geq 0$$

Subject to

$$y(0) = 1, y'(0) = 0, y''(0) = 2, y'''(0) = 0$$

For the existence of solution, let $\varepsilon = 0$ and this leads to

$$y(t) = 4\cos t - 3\cos\sqrt{2}t$$

Problem 5: Solve the equation

$$y'''' + 11y'' + 18y$$

Subject to the following conditions

$$y(0) = 1, y'(0) = 2, y''(0) = -3, y'''(0) = -2$$

Yields the analytical solution

$$\frac{6}{7}\cos\sqrt{2}x + \frac{8\sqrt{2}}{7}\sin\sqrt{2}x + \frac{1}{7}\cos 3x - \frac{2}{21}\sin 3x$$

TABLE 1 BELOW SHOWS THE CORRELATION BETWEEN THE THEORETICAL SOLUTION AND THE COMPUTED SOLUTION TO PROBLEM 1.

x	$y - Exact$	Approximated vales	Absolute Error
0.0	0.000000000000	0.000000000831	8.31E-09
0.1	0.200822385531	0.200822385201	3.3 E-10
0.2	0.407918161102	0.407918161112	1.0E-11
0.3	0.631568847162	0.631568847543	3.81E-10
0.4	0.887117420432	0.887117420701	2.69 E-10
0.5	1.195793458246	1.195793458114	1.32 E-10
0.6	1.585639601762	1.585639601201	5.61E-10
0.7	2.092578691621	2.092578691078	5.43E-10
0.8	2.761672360031	2.761672360642	6.11E-10
0.9	3.648635186243	3.648635186092	1.51E-10
1.0	4.821683805732	4.821683805334	3.98E-10

TABLE 2 SHOWS THE ERROR BETWEEN THE EXACT SOLUTION AND APPROXIMATE SOLUTION TO PROBLEM 2

x	$y - Exact$	Approximated vales	Absolute Error
0.0	1.000000000000	0.989657894678	1.03E-2
0.1	0.815594327751	0.815594327662	8.90E-11
0.2	0.664304475243	0.664304475903	6.60E-10
0.3	0.548381476253	0.548381476867	6.14E-10
0.4	0.469770462337	0.469770462224	1.13E-10
0.5	0.430657998403	0.430657998703	3.00E-10
0.6	0.433908683824	0.433908683110	7.14E-10
0.7	0.483439696442	0.483439696467	2.50E-11
0.8	0.584571574762	0.584571574795	3.30E-11
0.9	0.744386468127	0.744386468106	2.10E-11
1.0	0.972120963481	0.972120963486	5.00E-12

TABLE 3 CONTAINS THE ABSOLUTE ERROR BETWEEN THE ANALYTICAL SOLUTION AND THE APPROXIMATE SOLUTION TO PROBLEM 3

x	$y - Exact$	APPROXIMATE VALES	ABSOLUTE ERROR
0.0	-1.000000000000	-1.000000006707	6.707E-9
0.1	-0.814363935483	-0.814363935124	7.06E-10
0.2	-0.655146741872	-0.655146741045	4.0827E-8
0.3	-0.519385966634	-0.519385966115	5.19E-10
0.4	-0.404746096572	-0.404746096047	5.25E-10
0.5	0.019182261522	0.019182261453	6.90E-11
0.6	0.085745650385	0.085745650772	3.87E-10
0.7	0.127044695400	0.127044695952	5.52E-10
0.8	0.146553812700	0.146553812105	5.95E-10
0.9	0.146044014321	0.146044014663	3.42E-10
1.0	0.125841318001	0.125841318211	2.10E-10

TABLE 4 SHOWS THE COMPARISON BETWEEN THE ANALYTICAL SOLUTION AND THE APPROXIMATE SOLUTION TO PROBLEM 4

t	Y - Exact	Approximate values	Absolute Error
0.0	1.000000000000	1.000000000112	1.12E-10
0.1	1.275049876012	1.275049876301	2.89E-10
0.2	1.500531082003	1.500531082442	4.12E-10
0.3	1.677236278000	1.677236278223	2.23E-10
0.4	1.806347102000	1.806347102104	1.04E-10
0.5	1.889423330012	1.889423330931	9.19E-10
0.6	1.928390212101	1.928390212872	7.71E-10
0.7	1.925520648512	1.925520648254	2.58E-10
0.8	1.883414446102	1.883414446331	2.29E-10
0.9	1.804974212312	1.804974212671	3.59E-10
1.0	1.693378138503	1.693378138736	2.33E-10

TABLE 5 SHOWS THE COMPARISON BETWEEN THE ANALYTICAL SOLUTION AND THE APPROXIMATE SOLUTION TO PROBLEM 5

x	Y -Exact	Approximate Solution	Absolute Error
0.00	0.834550191422	0.834550191114	3.08E-10
0.1	0.229588862841	0.229588862091	7.50E-10
0.2	-0.084415637483	-0.084415637331	1.52E-10
0.3	-0.294341996764	-0.294341996783	1.90E-11
0.4	-0.444924148782	-0.444924148301	4.81E-10
0.5	-0.558898279431	-0.558898279442	1.10E-11
0.6	-0.648507179134	-0.648507179912	7.78E-10
0.7	-0.719912514856	-0.719912514084	7.72E-10
0.8	-0.775581917536	-0.775581917501	3.50E-11
0.9	-0.816110255482	-0.816110255746	2.64E-10
1.0	-0.841896716794	-0.841896716345	4.49E-10

DISCUSSION OF RESULTS

From Table 1, it was evident that the absolute error between the exact solution and approximate solution is highly infinitesimal. This shows that there is high correlation between the approximate solution and the exact solution. As a result of this, it points out that the method can be used to find solution to any linear fourth order ordinary differential equation.

From the observations in Table 2, there is little or no difference between the approximate solution and the true solution. This implies that the method derived can be appropriately used in finding solution to fourth order linear ordinary differential equations without necessarily resolving them into system of first order ordinary differential equations.

From the table 3 above, it was evident that the derived method compare favorably well with the analytic solution. This shows that the derived method can be used to evaluate the solution to any fourth order linear ordinary f=differential equations.

Table 4 shows the correlation between the approximate solution and the exact solution with respect to various values of the variable between$0\ and\ 1, with\ h = 0.1$. The result shows that there is little or no difference between the analytical solution and the approximate solution.

Table 5 present similar results when compare with the s=results in tables 1 to 4.

In general, it can be concluded that based on the examples given above that the derived method can be appropriately used for finding the solution to any fourth order linear ordinary differential equations. The beauty of the method is that it is self starting and does not require any additional starting value unlike the predictor – corrector method which required additional starting value and thereby altering the level of accuracy of the method. This is one of the beauties of the block method over and above the predictor – corrector method.

At this juncture based on the output of the numerical experiments as enumerated above, I am unequivocally recommending that the method derived is adequate to handle any fourth linear ordinary differential equations without going through the stress of resolving such an equation to system of first order ordinary differential equations.

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