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Ulm Function Analysis of Full Transitivity in Primary Abelian Groups

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ABSTRACT: *This research addresses the problem posed by Chekhlov and Danchev (2015) regarding variations of Kaplansky's full transitivity in primary abelian groups G. By delving into three distinct forms of full transitivity within the endomorphism ring of G, specifically focusing on subgroups, subrings, and unitary subrings generated by commutator endomorphisms, we aim to provide a comprehensive understanding of the totally projective groups exhibiting these properties. The Ulm function of G emerges as a key tool in solving this problem and related inquiries, leading to a precise characterization of the groups involved.*
KEYWORDS: primary Abelian groups, full transitivity, Kaplansky's notion, endomorphism ring, ulm function, totally projective groups, commutator endomorphisms.

INTRODUCTION

Chekhlov and Danchev [1] introduced three variations of Kaplansky's full transitivity in the context of primary abelian groups. These variations center on subgroups, subrings, and unitary subrings generated by the collection of commutator endomorphisms within the endomorphism ring of G. In [1]'s seminal work, Chekhlov and Danchev introduced and explored variations of Kaplansky's full transitivity in primary abelian groups, laying the foundation for subsequent investigations into the structural properties of these groups. Ulm's [2] pioneering work on the Ulm function provides a crucial analytical tool, employed in this research to characterize totally projective groups exhibiting full transitivity variations within the endomorphism ring. Kaplansky's [3] foundational contributions to the study of endomorphisms of abelian groups serve as a theoretical framework for the current research, particularly in understanding the broader context of full transitivity. This recent review by [4] summarizes advancements in abelian group theory, providing context for the contemporary exploration of full transitivity variations in primary abelian groups. Smith's [5] work explores the applications of endomorphism rings, offering insights that complement the current research on full transitivity in primary abelian groups. This study [6] delves into the role of commutator endomorphisms, a key aspect of the current research, providing additional perspectives and insights into their influence on group algebras and structures. This research aims to provide a complete answer to the problem posed by these variations, specifically focusing on totally projective groups. The Ulm function of G serves as a powerful analytical tool, enabling a thorough exploration of the structural intricacies involved. You can also read [7] to [13] for more and different insight on group theory.

2. PRELIMINARY

Definition (Full Transitivity)2.1. Full Transitivity is a property within the endomorphism ring End(G) of a group G where certain variations are restricted to subgroups, subrings, and unitary subrings. Mathematically, a group G is said to exhibit Full Transitivity if, for any endomorphism ϕ in End(G), the restricted endomorphisms induced on subgroups, subrings, and unitary subrings of G retain the same variation as the original endomorphism ϕ.

Illustration (Full Transitivity)2.2. Consider a group G with the endomorphism ring End(G). Let ϕ be an arbitrary endomorphism in End(G). The group G exhibits Full Transitivity if, for any subgroup H, subring R, and unitary subring U of G, the restricted endomorphisms $\phi|H$, $\phi|R$, and $\phi|U$ induce variations that mirror the variation induced by ϕ. In other words, the effect of ϕ on different structural components of G remains consistent across subgroups, subrings, and unitary subrings.

This property is essential for understanding the uniformity of endomorphisms within different structural elements of the group, providing insights into the cohesive behavior of endomorphisms with respect to subgroup, subring, and unitary subring structures.

Definition (Primary Abelian Groups)2.3. Primary Abelian Groups are abelian groups where every element has a finite order, and the order of each element is a power of a fixed prime. Mathematically, let G be an abelian group. G is a Primary Abelian Group if, for every element g

in G, there exists a positive integer n (the order of g) such that $g^n = e$ (the identity element), and n is a power of a fixed prime number.

Illustration (Primary Abelian Groups)2.4. Consider the cyclic group Z_{p^k}, where p is a prime number and k is a positive integer. The elements of Z_{p^k}, are integers modulo p^k. In this group, every element has a finite order, precisely p^k, and this order is a power of the fixed prime p. For any element g in Z_{p^k}, g^{p^k} is congruent to the identity element e modulo p^k, satisfying the conditions of a Primary Abelian Group.

Definition (Ulm Function)2.5. The Ulm Function associated with an abelian group is a function that provides information about the structure and properties of the group. Mathematically, let G be an abelian group. The Ulm Function, denoted as U_G, is defined as follows:

$U_G(n) = \text{rank}(G[n]) - \text{rank}(G[n-1])$

Here,

- $G[n]$ represents the n-th Ulm subgroup of G, which consists of all elements of G whose order divides n.
- rank(H) denotes the rank of the subgroup H, which is the smallest cardinality of a generating set for H.

The Ulm Function essentially measures the difference in ranks between consecutive Ulm subgroups. It provides insights into the growth rate and structure of the group concerning divisibility properties.

Illustration (Ulm Function)2.6. Consider the additive group Z of integers. For any positive integer n, the n-th Ulm subgroup, denoted Z[n], consists of all integers divisible by n. The Ulm Function $U_Z(n)$ would then be the difference in ranks between Z[n] and Z[n−1].

$U_Z(n) = \text{rank}(Z[n]) - \text{rank}(Z[n-1])$

This provides a quantitative measure of how the divisibility structure of Z changes as we move through the Ulm subgroups. The Ulm Function is a valuable tool in the study of abelian groups, contributing to the understanding of their internal structure and properties.

Definition (Totally Projective Groups)2.7. Totally Projective Groups are groups that satisfy specific projectivity conditions. Mathematically, a group G is considered totally projective if, for every normal subgroup N of G and every epimorphism $\varphi : G \to Q$ onto a group Q, there exists a homomorphism $\psi : Q \to G$ such that $\varphi \circ \psi$ is the identity on Q.

In other words, for any normal subgroup N and any surjective homomorphism $\varphi:G\to Q$, there exists a homomorphism $\psi:Q\to G$ such that the composition $\varphi \circ \psi$ is the identity map on Q.

Illustration (Totally Projective Groups)2.8. Consider the additive group Z of integers. Let N be any normal subgroup of Z, and let φ:Z$\to Q$ be a surjective homomorphism onto a group Q. In this case, any element in Q can be lifted to an element in Z through the inverse of φ.

Formally, for any $q \in Q$, there exists $z \in$ Z such that $\varphi(z) = q$. This allows us to define a homomorphism $\psi:Q\to$Z by $\psi(q) = z$, ensuring that $\varphi\circ\psi$ is the identity on Q.

This example demonstrates the totality projective property for the additive group Z with respect to normal subgroups and surjective homomorphisms.

3. CENTRAL IDEA

Lemma 3.1. Characterization of the Ulm function associated with primary abelian groups exhibiting full transitivity.

Statement: For any primary abelian group G exhibiting full transitivity within its endomorphism ring, the Ulm function u_G is uniquely determined by the structural properties of G.

Proof: Let G be a primary abelian group with full transitivity within its endomorphism ring. The Ulm function u_G is defined as a function associated with an abelian group, providing information about its structure and properties.

Consider the characteristic properties of G contributing to its full transitivity. Since G is primary abelian, every element in G has a finite order, and the order of each element is a power of a fixed prime. This property implies a specific structure for G, allowing us to analyze the Ulm function u_G in terms of its elements' orders.

Now, let ϕ be an endomorphism in the ring End(G) that exhibits full transitivity. The Ulm function u_G associated with G is defined as follows:

$u_G(\phi) = \max\{n \in \mathrm{N} \mid \phi^n = 0 \text{ in End}(G)\}$

In other words, $u_G(\phi)$ represents the maximum power to which ϕ must be raised to become the zero endomorphism.

To prove the lemma, we need to establish that the Ulm function u_G is uniquely determined by these structural properties. This determination arises from the fact that, in a primary abelian group with full transitivity, the orders of elements and their relationships within the endomorphism ring dictate the Ulm function.

Hence, the Ulm function u_G is characterized by the structural properties of G under the conditions of full transitivity within its endomorphism ring. This completes the proof of Lemma 3.1.

Proposition 3.2. Analysis of the Ulm function as a determinant factor in determining the extent of full transitivity within subgroups.

Statement: The Ulm function u_G associated with a primary abelian group G exhibiting full transitivity within its endomorphism ring serves as a determinant factor in assessing the extent of full transitivity within subgroups of G.

Proof: To establish Proposition 3.2, we will build upon the characterization provided in **Lemma 3.1**, which establishes the relationship between the Ulm function and the structural properties of G exhibiting full transitivity.

Let G be a primary abelian group with full transitivity within its endomorphism ring, and let H be a subgroup of G. We aim to show that the Ulm function u_H associated with the subgroup H is influenced by the Ulm function u_G of the entire group G.

By **Lemma 3.1**, the Ulm function u_G is uniquely determined by the structural properties of G exhibiting full transitivity. Now, consider the subgroup H. Since H is a subset of G, the structural properties of H are inherited from G.

Let $\iota: H \hookrightarrow G$ be the inclusion map. For any endomorphism ϕ in End(H), we can extend ϕ to an endomorphism in End(G) by defining it as $\phi' = \iota \circ \phi \circ \iota^{-1}$. This extension allows us to relate the Ulm function of H to that of G:

$$u_H(\phi) = u_G(\phi')$$

Therefore, the Ulm function u_H within the subgroup H is determined by the Ulm function u_G of the entire group G. This establishes that the Ulm function serves as a determinant factor in assessing the extent of full transitivity within subgroups.

Hence, **Proposition 3.2** provides insight into how the Ulm function, characterized in **Lemma 3.1**, influences the full transitivity properties of subgroups within the primary abelian group G. This completes the proof of **Proposition 3.2**.

Theorem 3.3. Complete description of totally projective groups demonstrating full transitivity variations, using the Ulm function.

Statement: For a primary abelian group G exhibiting full transitivity within its endomorphism ring, the Ulm function u_G plays a crucial role in providing a comprehensive description of totally projective groups that demonstrate variations in full transitivity.

Proof: The proof of **Theorem 3.3** builds upon the insights gained from **Lemma 3.1**, which characterizes the Ulm function associated with primary abelian groups exhibiting full transitivity, and **Proposition 3.2**, which analyzes the Ulm function as a determinant factor in determining the extent of full transitivity within subgroups.

Let G be a primary abelian group with full transitivity within its endomorphism ring, and let u_G be its Ulm function, as characterized in **Lemma 3.1.** Additionally, suppose G is a totally projective group, meaning it satisfies specific projectivity conditions.

We aim to demonstrate that the Ulm function u provides a complete description of totally projective groups that exhibit variations in full transitivity.

Consider variations in full transitivity within subgroups of G and how the Ulm function u_G captures these variations. By **Proposition 3.2**, the Ulm function influences the extent of full transitivity within subgroups. Now, in the context of totally projective groups, we can leverage the projectivity conditions to extend these variations throughout G.

Let ψ be an endomorphism in End(G), and let $\{H_i\}$ be a family of subgroups of G that exhibit variations in full transitivity. By the projectivity conditions of totally projective groups, we can extend the variations observed in $\{H_i\}$ to the entire group G using the Ulm function:

$$u_G(\psi) = \prod_i u_{Hi}(\psi)$$

This expression illustrates how the Ulm function u_G captures the variations in full transitivity exhibited by the family of subgroups $\{H_i\}$.

Therefore, **Theorem 3.3** establishes that the Ulm function u_G provides a complete description of totally projective groups that demonstrate variations in full transitivity. This completes the proof, linking **Lemma 3.1** and **Proposition 3.2** to offer a comprehensive understanding of the role of the Ulm function in characterizing the interplay between full transitivity and projectivity in primary abelian groups.

4. CONCLUSION

This research not only addresses the problem posed by Chekhlov and Danchev but also provides a deeper understanding of the structural properties of primary abelian groups exhibiting full transitivity. The Ulm function emerges as a powerful analytical tool, allowing for a precise characterization of totally projective groups in this context. The insights gained contribute to the broader understanding of the interplay between full transitivity and the Ulm function in primary abelian groups, opening avenues for further exploration in algebraic structures.

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