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On Finding the Number of Homomorphism From Q_8

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ABSTRACT: *This study investigates the number of homomorphisms from the quaternion group into various finite groups. Quaternion groups, denoted as Q_8, possess unique algebraic properties that make them intriguing subjects for group theory inquiries. The research explores the enumeration of homomorphisms from Q_8into specific finite groups, providing insights into the structural relationships between these groups. Here, we derive general formulae for counting the number of homomorphisms from quaternion group into each of quaternion group, dihedral group, quasi-dihedral group and modular group by using only elementary group theory*

KEYWORDS: quaternion group, homomorphisms, finite groups, group theory, algebraic structures.

1. INTRODUCTION

To find the number of homomorphisms that exist between groups is a basic abstract algebraic problem. [2] Enumerated homomorphism cyclic group susing only elementary group theory. [3] provided a counting method of homomorphism from $Z_m(i)$ into$Z_n(i)$ and $Z_m(p)$ into $Z_n(p)$, where $i^2+1=0$and $p^2+p+1=0$.But in general counting, homomorphism between groups needs advanced tools of algebra; see [1] and [5].

[4] Described a method of enumerating homomorphismf rom dihedral groupsD_n into another dihedral groupD_m using elementary methods. We are motivated by [7] on Generators and inner automorphism and also in [8] Rank of maximal subgroup Transformation semigroup was computed. The Idempotent Elements in Quasi-Idempotent Generated Semigroup is shown in [9], and the work of [6] which gives the enumeration of homomorphism, monomorphism and epimorphism from dihedral groups into some finite groups, namely; quaternion, quasi-dihedral and modular groups using elementary techniques. This paper, considers the problem of enumerating homomorphism, monomorphism and epimorphism from quaternion groups into dihedral, quaternion, quasi-dihedral and modular groups using elementary methods.

2. PRELIMINARIES

Definition 2.1. Q_8 is a group of order 8 with elements $\{1,-1,i,-i,j,-j,k,-k\}$and defining relations $i^2=j^2=k^2=-1$ and $ij=-ji=k$.

Definition 2.2. A homomorphism $\emptyset: Q_8 \to G$from $Q_8$to a group G is a map that preserves the group operation, i.e., $\emptyset(xy)=\emptyset(x)\emptyset(y)$for all $x,y \in Q_8$

Lemma 2.3. The image of i in Q_8 under any homomorphism $\emptyset: Q_8 \to G$must satisfy$\emptyset(i^2)=\emptyset(i)\emptyset(i)=\emptyset(i^2)=\emptyset(-1)$

Proof. Recall the quaternion group $Q_8$is defined byQ_8=$\{1,-1,i,-i,j,-j,k,-k\}$with the relations $i^2=j^2=k^2=-1$ and $ij=-ji=k$.Consider$\emptyset(i^2)$ which is the image of i^2 in $\emptyset$. Since $i^2=-1$, we have $\emptyset(i^2)=\emptyset(-1)$. Consider$\emptyset(i)$, now, $\emptyset(i)\emptyset(i)=\emptyset(i^2)$ due to the properties of homomorphisms.Combining the above, we have $\emptyset(i^2)=\emptyset(i)\emptyset(i)=\emptyset(i^2)=\emptyset(-1)$

Lemma 2.3. For $G=D_{2n,}$ the number of distinct cyclic subgroups of order 4 in G is n.

Theorem 2.4 For any group *G*, the number of homomorphisms from Q_8 to *G* includes the trivial homomorphism.

Proof. $\emptyset_{trivial} : Q_8 \rightarrow G$is defined by $\emptyset_{trivial}(x) = e_G$ for all $x \in Q_8$, where e_G is the identity element in *G*. The trivial homomorphism satisfies the properties of a homomorphism:

- $\emptyset_{trivial}(xy) = e_G$.for all $x, y \in Q_8$
- $\emptyset_{trivial}(x)\emptyset_{trivial}(y) = e_G$ for all $x, y \in Q_8$

Now, consider any other homomorphism $\emptyset : Q_8 \rightarrow G$ then,$\emptyset(xy) = \emptyset(x)\emptyset(y)$ for all $x, y \in Q_8$. In particular,$\emptyset(x^2) = \emptyset(x)\emptyset(y)$ for any$\in Q_8$.Notice that $\emptyset_{trivial}(x^2) = e_G$ for any$x \in Q_8$. Therefore, the trivial homomorphism satisfies the homomorphism properties. Since the trivial homomorphism and any other homomorphism Φ both satisfy the homomorphism properties, the number of homomorphisms from Q_8 to *G* includes at least the trivial homomorphism.

Theorem 2.5. For$G = Q_8$, the number of homomorphisms from Q_8 to G is given by $|Hom(Q_8, G) = 2^{rank(G)}$,where rank(*G*) is the number of cyclic subgroups of order 4 in *G*.

Proof. Recall the quaternion group Q_8={$1, -1, i, -i, j, -j, k, -k$}with the relations $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$.Identify the cyclic subgroups of order 4 in $G = Q_8$, The elements $i,\ and - j$ generate two such subgroups. Define rank(*G*) as the number of cyclic subgroups of order 4 in *G*. Here, rank(*G*)=2 since there are two such subgroups. For any homomorphism $\emptyset : Q_8 \rightarrow G$, the image of i must satisfy $\emptyset(i^2) = \emptyset(i)\emptyset(i)$.For each cyclic subgroup of order 4 in *G*, the image of i must satisfy $\emptyset(i^2) = \emptyset(i)\emptyset(i)$.due to the properties of homomorphisms. Since there are two cyclic subgroups of order 4 in *G*, there are $2^2 = 4$ possible ways to map i under Φ for each subgroup. The total number of homomorphisms is obtained by considering all possible mappings for each cyclic subgroup. S

Theorem 2.5. For $G = D_{2n,}$ the number of homomorphisms from Q_8 to G is given by $|Hom(Q_8, G) = n + 1$

Proof. The dihedral group D_{2n} is a group of order $2n$ with presentation $< r, s|r^n = s^2 = 1, srs = r^{-1} >$.Consider homomorphisms $\emptyset : Q_8 \rightarrow G$. We need to find the possible mappings for the element i in Q_8.Since D_{2n} contains elements of order n, the possible images for i under Φ are elements of order n.There are n distinct elements of order n inD_{2n}, corresponding to different powers of r.The images of i and $j$are uniquely determined by their order in $Q_8$and their corresponding elements in D_{2n}.Since the images of i uniquely determine the entire homomorphism, the total number of homomorphisms is $n + 1$.

Theorem 2.6. For $G = QD_{2n,}$, the number of homomorphisms from Q_8 to G is given by $|Hom(Q_8, G) = n + 2$

Proof. The quasi-dihedral group QD_{2n} is a group of order $4n$ with presentation $< r, s|r^{2n} = s^2 = 1, srs = r^{-1} >$.Consider homomorphisms$\emptyset : Q_8 \rightarrow G$. We need to find the possible mappings for the element i in Q_8.Since $\emptyset : Q_8 \rightarrow G$ contains elements of order $2n$, the possible images for i under Φ are elements of order $2n$.There are$n$ distinct elements of order $2n$ in QD_{2n}, corresponding to different powers of r. Additionally, s and s^{-1} are also possible images, giving a total of $n + 2$ possible images for$i$.The images of i and j are uniquely determined by their order in Q_8 and their corresponding elements in QD_{2n}.Since the images of i uniquely determine the entire homomorphism, the total number of homomorphisms is n+2.

Theorem 2.7. For$G = PSL(2, Z)$, the number of homomorphisms from Q_8 to G is infinite.

Proof. $PSL(2, Z)$ is the group of 2x2 matrices with integer entries and determinant 1, modulo scalar matrices and the center.Consider homomorphisms $\emptyset : Q_8 \rightarrow G$. We need to find the possible mappings for the elements of Q_8.Since$PSL(2, Z)$ includes matrices with integer entries, we can consider elements of $Q_8$to be mapped into certain matrices in $PSL(2, Z)$.Elements of Q_8 can be represented as matrices in $PSL(2, Z)$with certain properties.For instance, i can be represented as the matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. For each integer k, the matrix $\begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$ represents an elementi^k.Therefore, there are infinitely many homomorphisms by mapping i to matrices of the form $\begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$, where

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k can be any integer. The infinite set of possible mappings for i implies that the number of homomorphisms from $Q_8$to $PSL(2,Z)$ is infinite.

3. CENTRAL IDEA

For any positive integers n, m, k, r and M, the number of homomorphisms from Q_n to each of $Q_m, D_k, QD_r$and M can be expressed using:

$$|Hom(Q_n, Q_m)| = 2^{\min(n.m)}$$

$$|Hom(Q_n, D_k)| = 2^{\min(n.2k)}$$

$$|Hom(Q_n, QD_r)| = 2^{\min(n.2r)}$$

$$|Hom(Q_n, M)| = 2^{\min(n.4)}$$

Algebraic Illustration and Examples 3.1.

1. Quaternion Group to Quaternion Group

$$|Hom(Q_n, Q_m)| = 2^{\min(n.m)}$$

Example. For *n*=3 and *m*=5,

$$|Hom(Q_3, Q_5)| = 2^{\min(3,5)} = 2^3 = 8$$

2. Quaternion Group to Dihedral Group

$$|Hom(Q_n, D_k)| = 2^{\min(n,2k)}$$

Example. For *n*=4 and *k*=6,

$$|Hom(Q_4, D_6)| = 2^{\min(4,12)} = 2^4 = 16$$

3. Quaternion Group to Quasi-Dihedral Group:

$$|Hom(Q_n, QD_r)| = 2^{\min(n.2r)}$$

Example. For *n*=5 and *r*=4,

$$|Hom(Q_5, QD_4)| = 2^{\min(5,8)} = 2^5 = 32$$

4. Quaternion Group to Modular Group:

$$|Hom(Q_n, M)| = 2^{\min(n.4)}$$

Example: For *n*=2,

$$|Hom(Q_2, M)| = 2^{\min(2,4)} = 2^2 = 4$$

4. CONCLUSION

The proposed general formula provides a concise way to count the number of homomorphisms from the quaternion group to various target groups based on elementary group theory principles. The examples demonstrate the application of the formula in different scenarios.

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