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Autocovariances and Autocorrelation Properties of Diagonal Vector Autoregressive and Multivariate Autoregressive Distributed Lag Models

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Abstract: *The primary aim of this study was to conduct a comparative analysis of the performance of parsimonious models, specifically the Diagonal Vector Autoregressive (VAR) and Multivariate Autoregressive Distributed Lag (MARDL) Models, using their respective Autocovariance and Autocorrelation properties. This comparison was driven by the imposition of restrictions on parameters within the coefficient matrices, specifically limiting them to diagonal elements. To assess the efficacy of these novel multivariate lag models, we utilized data derived from key macroeconomic variables, including Nigeria's Gross Domestic Product (GDP), Crude Oil Petroleum (C/PET), Agriculture (AGRIC), and Telecommunication (TELECOM). The data was subjected to first-order differencing of the logarithm of the series to ensure stationarity. Subsequently, the models were estimated, and Autocovariances and Autocorrelations of the processes were derived for the analysis. The empirical findings revealed notable patterns, particularly the direct converse autocorrelation observed in both VAR and MARDL models. The negative autocorrelation identified in the macroeconomic variables suggests that periods of economic expansion were succeeded by contractions and vice versa. This implies a complementary relationship between the two models in effectively capturing the dynamics of multivariate lag variables. In conclusion, our study underscores the significance of considering the Diagonal Vector Autoregressive and Multivariate Autoregressive Distributed Lag Models with restricted parameters in the diagonal elements when modelling multivariate lag variables. These findings contribute to a nuanced understanding of the interplay between economic variables and provide valuable insights for researchers and practitioners in the field*.

Keywords: variances, autocovariance, autocorrelation, upper and lower diagonal VAR models, upper and lower diagonal MARDL models.

INTRODUCTION

VAR models extend univariate time series models by incorporating a response variable as a function of its lag terms, emphasizing a feed-forward and feed-back mechanism. Each response variable in Vector Autoregressive Models is a linear combination of its lag terms, predictors, and an error term, mirroring a multiple linear regression model. This characteristic, capturing the interdependence and dynamics among response and predictor variables, contributes to the versatility of VAR models in predicting economic and financial time series dynamics. The Vector Autoregressive model is one of the models for the analysis of multivariate time series. VAR models gained prominence in the field of economics following the work of Sims (1980) and represent a logical expansion of the univariate autoregressive model. The VAR model is useful for describing the dynamic behaviour of time series data. There are many studies about modelling financial time series with VAR models one of which includes Chatfield (1996). Vector autoregressive (VAR) models are one of the econometric models which provide a simple tool for characterising the dynamic interaction of the data, which can be displayed either by their autocovariance and autocorrelation functions or by their impulse response functions. The latter may be sensitive to the validity of a set of assumptions used to identify particular structural shocks in the data. Bernanke and Mihov (1998), Christiano, Eichenbaum, and Evans (1999). To avoid the need to identify structural shocks, McCallum (1999) advocated the use of autocovariance and autocorrelation functions as a more appropriate device for confronting economic models with the data. The autocovariance and autocorrelation functions are computed from coefficients of VAR models which are estimates of a data set. Coenen, Günter (2000), in his work "Asymptotic confidence bands for the estimated autocovariance and autocorrelation functions of vector autoregressive models" provided formulae for computing the asymptotic standard errors of the estimated autocovariance and autocorrelation functions of stable VAR models, these he stipulated can be used to construct asymptotic confidence bands, where the sample autocovariance and the sample autocorrelations are asymptotically normal.

Hannan (1970) and Anderson (1971) indicated that the test relying on sample autocorrelations is easily conducted, with standard errors approximately equal to $1/\sqrt{T}$. However, Dufour and Roy (1985) demonstrated that these tests may exhibit a lower frequency of rejecting the null hypothesis than expected. Coenen and Günter (2000) established asymptotic normality for estimating autocovariance and autocorrelation functions under specific conditions, suggesting that data from a VAR process might be rejected if the process is inconsistent with its nominal size. VARMA models are discussed in works like Reinsel (1993), while the SCA Statistical System Liu and Gregory (1994). is also relevant to this context. provides the available software for forecasting and time series analysis using VARMA models. SCA only uses relatively simple and effective tools to determine the order of a pure vector moving average (VMA) or a pure vector autoregressive (VAR) model (Tiao and Box, 1981). Also, Pestano and González (2004) provide new answers to difficult questions on identifiability, minimality and exchangeability. They mention several possible ways to analyse the problems through the use of the autocovariance matrices, the coefficients of the infinite VAR form, the coefficients of the infinite VMA form. Furthermore, Hamilton (1994), Lutkepohl (2005), Amisano and Giannini (1997), and Stock and Watson (2001) offer a comprehensive, non-technical exploration of vector autoregressions and their significance in macroeconomics. Becketti (2013) serves as a valuable resource for an introductory understanding of VAR analysis, with a focus on practical implementation. In the absence of constraints on the coefficients, the VAR(p) can be viewed as a seemingly unrelated regression model, featuring identical explanatory variables in each equation. As elucidated in Lutkepohl (2005) and Greene (2008), conducting linear regression on each equation yields maximum likelihood estimates of the coefficients. The authoritative technical reference for VAR models is Lutkepohl (1991), and updated surveys on VAR techniques can be found in the works of Watson (1994), Lutkepohl (1999), and Waggoner and Zha (1999). Noteworthy applications of VAR models to financial data are detailed in the contributions of Campbell, Lo, and MacKinlay (1997) and Tsay (2001).

Multivariate Autoregressive Distributed Lag Models (MARDLM) constitute a framework designed for the analysis of multiple responses. These models hinge on both lagged and non-lagged terms of predictor variables, along with exclusively lagged terms of the response. In the context of multivariate time series, each temporal variable is expressed as a linear combination of its lagged terms and those of other variables. Usoro (2019) innovatively introduced MARDLM by amalgamating Multivariate Linear Regression Models (MLRM) and Vector Autoregressive Models (VARM). The Multivariate Linear Regression Models delineate the linear relationship between the present response time and predictor variables, while Vector Autoregressive Models are renowned for modelling various time series characterized by autoregressive processes. Multivariate Autoregressive Distributed Lag Models were developed to represent the contemporary influence of predictor factors in a multiple linear relationship between the response and a set of predictor variables. What sets MARDLM apart from VARM is its incorporation of the present time of the predictor variable in each relationship, whereas VARM confines independent variables to predictor lagged terms. This distinction is rooted in the inherent causal relationship between the current time of predictor variables and the response, underscoring the practical significance of MARDLM in capturing real-world dynamics. Usoro (2020), On the Stationarity of Multivariate Time Series applied cross-autocorrelation and cross-autocorrelation matrix to ascertain the stationarity. The investigation unveiled that in instances of instability within a multivariate time series, the stability of the multivariate process is influenced through partial stationarity. Usoro (2019) developed the MARDL model by combing the multiple linear regression model with the VAR model, Also, Usoro and Udoh (2021) applied the MARDL model in the modelling of Nigerian Gross Domestic Product and other macro-economic variables. The result showed a complementary role of thus models to each other.

Autocovariance and Autocorrelation Application in Time Series.

Bouri et al. (2017) delved into autocorrelation and volatility patterns within the cryptocurrency market. The authors employed GARCH models to capture time-varying volatility. Their findings indicated a significant role of autocorrelation in predicting future price movements. Anselin (2002) proposed an autocovariance-based approach to estimate spatial dependence in econometric models. The study demonstrated the effectiveness of this method in capturing spatial patterns and dependencies, providing a valuable tool for spatial econometric analysis. Jiang et al. (2015) investigated autocorrelation and volatility clustering phenomena in Chinese stock returns. By utilizing ARCH models, the study identified significant autocorrelation and clustering patterns, thereby contributing to the understanding of market dynamics. Dogan and Deger (2017) examined the autocorrelation and cross-correlation between electricity consumption and economic indicators in Turkey. Through time-series analysis, the study revealed dynamic relationships that could inform energy policy decisions. Omay and Yuksel (2013) analyzed the autocorrelation structure of exchange rate returns, focusing on the Turkish lira. The study utilized ARIMA models and identified unique autocorrelation patterns, contributing to the understanding of currency market dynamics.

Vázquez (2006) employed a regime-switching framework to analyze autocorrelation in credit spreads. The study provided insights into the dynamic behavior of credit spreads, with implications for risk management and financial decision-making. Li and Hamori (2013) investigated the autocorrelation and cross-correlation between global oil prices and stock returns. Through time-series analysis, the study revealed significant relationships with implications for portfolio management and investment strategies. Doz (2012) proposed autocovariance-based estimators for large dynamic factor models, aiming to enhance the efficiency of estimation. The study introduced novel techniques to handle large datasets, offering improved statistical inference in factor modeling. Cuestas and Tang (2018) examined autocorrelation in the stock returns of systemically important banks in the globalized financial system. Using time-series techniques, the study uncovered autocorrelation patterns, contributing to the understanding of interconnectedness in the banking sector.

Wang et al. (2018) presented a comprehensive review analyzing autocorrelation in financial time series data. The authors summarized various methods used for financial autocorrelation analysis and discussed their applications. Gelfand et al. (2012) introduced a Bayesian approach to model spatial autocorrelation in ecological data. The study illustrated their method with an application to species distribution modeling. Smith et al. (2016) investigated autocorrelation patterns in climate data and their implications for trend detection. The authors proposed a methodology to account for autocorrelation when analyzing long-term climate trends. Li et al. (2019) explored autocovariance analysis for detecting sleep stages based on neurophysiological signals. The authors presented a novel approach and validated it using polysomnography data. Zhang et al. (2017) applied autocorrelation analysis to DNA sequence patterns to predict gene expression levels. The authors demonstrated the effectiveness of their approach through computational experiments.

General VAR Model

Definition

Let $\underline{Z}_t = (Z_{1t}, Z_{2t}, \dots, Z_{mt})^I$ be the vector of response time variables, $\underline{\emptyset} = \left(\emptyset_{k.ij}\right)$ is the vector of coefficients, $\underline{Z}_{t-k} = (Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})^I$ be defined as the vector of the predictive lag time variables, $\underline{\delta} = (\delta_1, \delta_2, \dots, \delta_m)^I$ is the vector of constants and $\underline{w_t} = (w_{t1}, w_{2t}, \dots, w_{mt})^I$ is the vector of error terms associated with the response time variables. The Vector Autoregressive Model is presented in the form,

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & \emptyset_{1.12} & \cdots & \emptyset_{1.1n} \\ \emptyset_{1.21} & \emptyset_{1.22} & \cdots & \emptyset_{1.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{1.m1} & \emptyset_{1.m2} & \dots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix} + \begin{pmatrix} \emptyset_{2.11} & \emptyset_{2.12} & \cdots & \emptyset_{2.1n} \\ \emptyset_{2.21} & \emptyset_{2.22} & \cdots & \emptyset_{2.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{2.m1} & \emptyset_{2.m2} & \dots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix} + \quad \cdots \quad + \begin{pmatrix} \emptyset_{p.11} & \emptyset_{p.12} & \cdots & \emptyset_{p.1n} \\ \emptyset_{p.21} & \emptyset_{p.22} & \cdots & \emptyset_{p.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{p.m1} & \emptyset_{p.m2} & \dots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} + \begin{pmatrix} w_{1t} \\ w_{2t} \\ \vdots \\ w_{mt} \end{pmatrix} \qquad (1)$$

The more simplified form of Equation (1) by Gujarati and Porter (2009) is

$$Z_{it} = \delta_i + \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.ij} Z_{jt-k} + w_{it}, \; i = 1, \dots, m \qquad (2)$$

The set of General Vector Autoregressive Models Is defined below,

$$Z_{it} = \begin{cases} \delta_1 + \emptyset_{k.1j} Z_{jt-k} + w_{1t}, \; i = 1; j = 1, \dots, n; k = 1, \dots, p \\ \delta_2 + \emptyset_{k.2j} Z_{jt-k} + w_{2t}, \; i = 2; j = 1, \dots, n; k = 1, \dots, p \\ \delta_3 + \emptyset_{k.3j} Z_{jt-k} + w_{3t}, i = 3; j = 1, \dots, n; k = 1, \dots, p \\ \vdots \\ \delta_m + \emptyset_{k.mj} Z_{jt-k} + w_{mt}, i = m; j = n; k = 1, \dots, p \end{cases} \qquad (3)$$

Autocovariances of the General VAR Models.

Autocovariances of $Z_{1t} Z_{1t-l}$

$for\ \delta_1 = 0, i = 1,$ multiply Equation (3) by Z_{1t-l} and take the expectations.

$$\begin{aligned} E(Z_{1t} Z_{1t-l}) = {} & \varphi_{1.11} E(Z_{1t-l} Z_{1t-1}) + \varphi_{1.12} E(Z_{1t-l} Z_{2t-1}) + \cdots + \varphi_{1.1n} E(Z_{1t-l} Z_{nt-1} \\ & + \varphi_{2.11} E(Z_{1t-l} Z_{1t-2}) + \varphi_{2.12} E(Z_{1t-l} Z_{2t-2}) + \cdots + \varphi_{2.1n} E(Z_{1t-l} Z_{nt-2}) + \cdots \\ & + \varphi_{p.11} E(Z_{1t-l} Z_{1t-p}) + \varphi_{p.12} E(Z_{1t-l} Z_{2t-p}) + \cdots + \varphi_{p.1n} E(Z_{1t-l} Z_{nt-p}) \\ & + E(Z_{1t-l} \epsilon_{1t}) \end{aligned}$$

$$E(Z_{1t} Z_{1t-l}) = \xi_{1t,1t(l)}$$

$$\xi_{1t,1t(l)} = \varphi_{1.11}\xi_{1t(l-1),1t} + \varphi_{1.12}\xi_{1t(l-1),2t} + \cdots + \varphi_{1.1n}\xi_{1t(l-1),nt} + \varphi_{2.11}\xi_{1t(l-2),1t} + \varphi_{2.12}\xi_{1t(l-2),2t} + \cdots + \varphi_{2.1n}\xi_{1t(l-2),nt} + \cdots + \varphi_{p.11}\xi_{1t,1t(l-p)} + \varphi_{p.12}\xi_{1t,2t(pl-p)} + \cdots + \varphi_{p.1n}\xi_{1t,nt(l-p)}$$

$E(Z_{1t-l}\epsilon_{1t}) = 0$ (uncorrelated stationary process)

$$\xi_{1t,1t(l)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\xi_{1t(l-k),jt} \quad ; \qquad l = 1,2,3,\ldots \tag{4}$$

Autocovariance of $Z_{2t}Z_{2t-l}$

$for\ \delta_2 = 0, i = 2,$ multiply Equation (3) by $Z_{2\,t-l}$ and take the expectations.

$$E(Z_{2t}Z_{2t-l}) = \varphi_{1.21}E(Z_{2t-l}Z_{1t-1}) + \varphi_{1.22}E(Z_{2t-l}Z_{2t-1}) + \cdots + \varphi_{1.2n}E(Z_{2t-l}Z_{nt-1} + \varphi_{2.21}E(Z_{2t-l}Z_{1t-2}) + \varphi_{2.22}E(Z_{2t-l}Z_{2t-2}) + \cdots + \varphi_{2.2n}E(Z_{2t-l}Z_{nt-2}) + \cdots + \varphi_{p.21}E(Z_{2t-l}Z_{1t-p}) + \varphi_{p.22}E(Z_{2t-l}Z_{2t-p}) + \cdots + \varphi_{p.2n}E(Z_{2t-l}Z_{nt-p}) + E(Z_{2t-l}\epsilon_{2t})$$

$$E(Z_{2t}Z_{2t-l}) = \xi_{2t,2t(l)}$$

$$\xi_{2t,2t(l)} = \varphi_{1.21}\xi_{2t(l-1),1t} + \varphi_{1.22}\xi_{2t(l-1),2t} + \cdots + \varphi_{1.2n}\xi_{2t(l-1),nt} + \varphi_{2.21}\xi_{2t(l-2),1t} + \varphi_{2.22}\xi_{2t(l-2),2t} + \cdots + \varphi_{2.2n}\xi_{2t(l-2),nt} + \cdots + \varphi_{p.21}\xi_{2t(l-p),1t} + \varphi_{p.22}\xi_{2t(l-p),2t} + \cdots + \varphi_{p.2n}\xi_{2t(l-p),nt}$$

$E(Z_{2t-l}\epsilon_{2t}) = 0$ (uncorrelated processes)

$$\xi_{2t,2t(l)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\xi_{2t(l-k),jt} \quad ; \qquad l = 1,2,3,\ldots \tag{5}$$

$\vdots \qquad \vdots \qquad \vdots$

Autocovariance of $Z_{mt}Z_{mt-l}$

$for\ \delta_3 = 0, i = 3,$ multiply Equation (3) by Z_{mt-l} and take the expectations.

$$E(Z_{mt}Z_{mt-l}) = \varphi_{1.m1}E(Z_{mt-l}Z_{1t-1}) + \varphi_{1.m2}E(Z_{mt-l}Z_{2t-1}) + \cdots + \varphi_{1.mn}E(Z_{mt-l}Z_{nt-1} + \varphi_{2.m1}E(Z_{mt-l}Z_{1t-2}) + \varphi_{2.m2}E(Z_{mt-l}Z_{2t-2}) + \cdots + \varphi_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \cdots + \varphi_{p.m1}E(Z_{mt-l}Z_{1t-p}) + \varphi_{p.m2}E(Z_{mt-l}Z_{2t-p}) + \cdots + \varphi_{p.mn}E(Z_{mt-l}Z_{nt-p}) + E(Z_{mt-l}\epsilon_{mt})$$

$$E(Z_{mt}Z_{mt-l}) = \xi_{mt,mt(l)}$$

$$\xi_{mt,mt(l)} = \varphi_{1.m1}\xi_{mt(l-1),1t} + \varphi_{1.m2}\xi_{mt(l-1),2t} + \cdots + \varphi_{1.mn}\xi_{mt(l-1),nt} + \varphi_{2.m1}\xi_{mt(l-2),1t} + \varphi_{2.m2}\xi_{mt(l-2),2t} + \cdots + \varphi_{2.mn}\xi_{mt(l-p),nt} + \cdots + \varphi_{p.m1}\xi_{mt(l-p),1t} + \varphi_{p.m2}\xi_{mt(l-p),2t} + \cdots + \varphi_{p.mn}\xi_{mt(l-p),nt}$$

$E(Z_{mt-l}\epsilon_{mt}) = 0$ (uncorrelated processes)

$$\xi_{mt,mt(l)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\xi_{mt(l-k),jt} \quad ; \quad l = 1,2,3,\ldots \tag{6}$$

Autocorrelations of the General VAR Models.
The Autocorrelation of Z_{1t} and Z_{1t-l}

$\rho_{1t,1t(l)} = \dfrac{\xi_{1t,1t(l)}}{\xi_{1t,1t}}$,

Where $\xi_{1t,1t}$ is given as $\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.1j}\xi_{1t,jt(k)} + \sigma_{w_{1t}}^2$, Usoro and Udoh (2023)

Therefore, from Equation (4)

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\xi_{1t(l-k),jt}}{\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.1j}\xi_{1t,jt(k)} + \sigma_{w_{1t}}^2} & , l = 1, 2, 3 \ldots \end{cases} \tag{7}$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} and $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} and Z_{2t-l}

$\rho_{2t,2t(l)} = \dfrac{\xi_{2t,2t(l)}}{\xi_{2t,2t}}$,

Where $\xi_{2t,2t}$ is given as $\sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma_{w_{2t}}^2$; Usoro and Udoh (2023)

Therefore, from Equation (5)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & , I = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\xi_{2t(l-k),jt}}{\sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma_{w_{2t}}^2} & , l = 1, 2, 3 \ldots \end{cases} \tag{8}$$

$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} and $\xi_{2t(l)}$

$\vdots \qquad \vdots \qquad \vdots$

The Autocorrelation of Z_{mt} and Z_{mt-l}

$\rho_{mt,mt(l)} = \dfrac{\xi_{mt,mt(l)}}{\xi_{mt,mt}}$,

Where $\xi_{mt,mt}$ is given as $\sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt,nt(k)} + \sigma_{w_{mt}}^2$; Usoro and Udoh (2023)

Therefore, from Equation (6)

$$\rho_{mt,mt(l)} = \begin{cases} 1 & ,l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\xi_{mt(l-k),jt}}{\sum_{k=1}^{p}\emptyset_{k.mn}\xi_{mt,nt(k)} + \sigma_{w_{mt}}^2} & ,l = 1, 2, 3 \dots \end{cases} \quad (9)$$

$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

Upper Diagonal VAR Models and Their Properties

Model Derivation

This section considers upper diagonal VAR models from the general form.
Give the following equation matrix of upper diagonal coefficients,

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & \emptyset_{1.12} & \cdots & \emptyset_{1.1n} \\ 0 & \emptyset_{1.22} & \dots & \emptyset_{1.2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix} + \begin{pmatrix} \emptyset_{2.11} & \emptyset_{2.12} & \cdots & \emptyset_{2.1n} \\ 0 & \emptyset_{2.22} & \dots & \emptyset_{2.2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix} + \dots + \begin{pmatrix} \emptyset_{p.11} & \emptyset_{p.12} & \cdots & \emptyset_{p.1n} \\ 0 & \emptyset_{p.22} & \dots & \emptyset_{p.2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} + \begin{pmatrix} w_{1t} \\ w_{2t} \\ \vdots \\ w_{mt} \end{pmatrix} \quad (10)$$

Usoro and Udoh (2023) reduced the above set of diagonal models to the form,

$$Z_{it} = \begin{cases} \delta_1 + \emptyset_{k.1j}Z_{jt-k} + w_{1t}, & i = 1; j = 1, \dots, n; k = 1, \dots, p \\ \delta_2 + \emptyset_{k.2j}Z_{jt-k} + w_{2t}, & i = 2; j = 2, \dots, n; k = 1, \dots, p \\ \delta_3 + \emptyset_{k.3j}Z_{jt-k} + w_{3t}, & i = 3; j = 3, \dots, n; k = 1, \dots, p \\ \vdots \\ \delta_m + \emptyset_{k.mj}Z_{jt-k} + w_{mt}, & i = m; j = n; k = 1, \dots, p \end{cases} \quad (11)$$

Equation (11) defines a set of Upper Diagonal Vector Autoregressive Models.

Autocovariances of Upper Diagonal VAR Models

Autocovariances of $Z_{1t}Z_{1t-l}$

$for\ \delta_1 = 0, i = 1,$ multiply Equation (11) by Z_{1t-l} and take the expectations.

$$E(Z_{1t}Z_{1t-l}) = E[Z_{1t-l}(\emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Z_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}Z_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t})]$$

$$E(Z_{1t}Z_{1t-l}) = \emptyset_{1.11}E(Z_{1t-1}Z_{1t-l}) + \emptyset_{1.12}E(Z_{1t-l}Z_{2t-1}) + \cdots + \emptyset_{1.1n}E(Z_{1t-l}Z_{nt-1}) + \emptyset_{2.11}E(Z_{1t-l}Z_{1t-2}) + \emptyset_{2.12}E(Z_{1t-l}Z_{2t-2}) + \cdots + \emptyset_{2.1n}E(Z_{1t-l}Z_{nt-2}) + \cdots + \emptyset_{p.11}E(Z_{1t-l}Z_{1t-p}) + \emptyset_{p.12}E(Z_{1t-l}Z_{2t-p}) + \cdots + \emptyset_{p.1n}E(Z_{1t-l}Z_{nt-p}) + E(Z_{1t-l}\varepsilon_{1t})$$

$E(Z_{1t}Z_{1t-l}) = \xi_{1t,1t(l)}$

$$\xi_{1t,1t(l)} = \emptyset_{1.11}\xi_{1t(l-1),1t} + \emptyset_{1.12}\xi_{1t(l-1),2t} + \cdots + \emptyset_{1.1n}\xi_{1t(l-1),nt} + \emptyset_{2.11}\xi_{1t(l-2),1t} + \emptyset_{2.12}\xi_{1t(l-2),2t} + \cdots + \emptyset_{2.1n}\xi_{1t(l-2),nt} + \cdots + \emptyset_{p.11}\xi_{1t(l-p),1t} + \emptyset_{p.12}\xi_{1t(l-p),2t} + \cdots + \emptyset_{p.1n}\xi_{1t(l-p),nt}$$

where, $E(Z_{1t-l}\varepsilon_{1t})$ = 0 (uncorrelated stationary processes)

$$\xi_{1t,1t(l)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.1j}\xi_{1t(l-k),jt}, l = 1,2,3, \ldots \quad (12)$$

Autocovariances of $Z_{2t}Z_{2t-l}$

$for\ \delta_2 = 0, \quad i = 2,$ multiply Equation (11) by Z_{2t-l} and take the expectations.

$$E(Z_{2t}Z_{2t-l}) = E[Z_{2t-l}(\emptyset_{1.22}Z_{2t-1} + \emptyset_{1.23}Z_{3t-1} + \cdots + \emptyset_{1.2n}Z_{nt-1} + \emptyset_{2.22}Z_{2t-2} + \emptyset_{2.23}Z_{3t-2} + \cdots + \emptyset_{2.2n}Z_{nt-2} + \cdots + \emptyset_{p.22}Z_{2t-p} + \emptyset_{p.23}Z_{3t-p} + \cdots + \emptyset_{p.2n}Z_{nt-p} + \varepsilon_{2t})]$$

$$E(Z_{2t}Z_{2t-l}) = \emptyset_{1.22}E(Z_{2t-l}Z_{2t-1}) + \emptyset_{1.23}E(Z_{2t-l}Z_{3t-1}) + \cdots + \emptyset_{1.2n}E(Z_{2t-l}Z_{nt-1}) + \emptyset_{2.22}E(Z_{2t-l}Z_{2t-2}) + \emptyset_{2.23}E(Z_{2t-l}Z_{3t-2}) + \cdots + \emptyset_{2.2n}E(Z_{2t-l}Z_{nt-2}) + \cdots + \emptyset_{p.22}E(Z_{2t-l}Z_{2t-p}) + \emptyset_{p.23}E(Z_{2t-l}Z_{3t-p}) + \cdots + \emptyset_{p.2n}E(Z_{2t-l}Z_{nt-p}) + E(Z_{2t-l}\varepsilon_{2t})$$

$E(Z_{2t}Z_{2t-l}) = \xi_{2t,2t(l)}$

$$\xi_{2t,2t(l)} = \emptyset_{1.22}\xi_{2t(l-1),2t} + \emptyset_{1.23}\xi_{2t(l-1),3t} + \cdots + \emptyset_{1.2n}\xi_{2t(l-1),nt} + \emptyset_{2.22}\xi_{2t(l-2),2t} + \emptyset_{2.23}\xi_{2t(l-2),3t} + \cdots + \emptyset_{2.2n}\xi_{2t(l-2),nt} + \cdots + \emptyset_{p.22}\xi_{2t(l-p),2t} + \emptyset_{p.23}\xi_{2t(l-p),3t} + \cdots + \emptyset_{p.2n}\xi_{2t(l-p),nt}$$

where, $E(Z_{2t-l}\varepsilon_{2t})$ = 0 (uncorrelated stationary process)

$$\xi_{2t,2t(l)} = \sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t(l-k),jt}\ l = 1,2,3, \ldots \quad (13)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Autocovariances of $Z_{mt}Z_{mt-l}$

$for\ \delta_m = 0, i = m$ multiply Equation (11) by Z_{mt-l} and take the expectations.

$$E(Z_{mt}Z_{mt-l}) = E[Z_{mt-l}(\emptyset_{1.mn}Z_{nt-1} + \emptyset_{2.mn}Z_{nt-2} + \cdots + \emptyset_{p.mn}Z_{nt-p} + \varepsilon_{mt})]$$

$$E(Z_{mt}Z_{mt-l}) = \emptyset_{1.mn}E(Z_{mt-l}Z_{nt-1}) + \emptyset_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \cdots + \emptyset_{p.mn}E(Z_{mt-l}Z_{nt-p}) + E(Z_{mt-l}\varepsilon_{mt})$$

$$E(Z_{mt}Z_{mt-l}) = \xi_{mt,mt(l)}$$

$$\xi_{mt,mt(l)} = \emptyset_{1.mn}\xi_{mt(l-1),nt} + \emptyset_{2.mn}\xi_{mt(l-2),nt} + \cdots + \emptyset_{p.mn}\xi_{mt(l-p),nt}$$

where, $E(Z_{mt-l}\varepsilon_{mt}) = 0$ (uncorrelated stationary process)

$$\xi_{mt,mt(l)} = \sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt(l-k),nt}\,, l = 1,2,3,\ldots \quad (14)$$

Autocorrelations of the Upper Diagonal VAR Models

The Autocorrelation of Z_{1t} and Z_{1t-l}

$$\rho_{1t,1t(l)} = \frac{\xi_{1t,1t(l)}}{\xi_{1t,1t}},$$

Where $\xi_{1t,1t}$ is given as $\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.1j}\xi_{1t,jt(k)} + \sigma^2_{w_{1t}}$, Usoro and Udoh (2023)

Therefore, from Equation (12)

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\xi_{1t(l-k),jt}}{\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.1j}\xi_{1t,jt(k)} + \sigma^2_{w_{1t}}} & , l = 1,2,3,\ldots \end{cases} \quad (15)$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} *and* $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} and Z_{2t-l}

$$\rho_{2t,2t(l)} = \frac{\xi_{2t,2t(l)}}{\xi_{2t,2t}},$$

Where $\xi_{2t,2t}$ is given as $\sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma^2_{w_{2t}}$, Usoro and Udoh (2023)

Therefore, from Equation (13)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\xi_{2t(l-k),jt}}{\sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma^2_{w_{2t}}} & , l = 1,2,3,\ldots \end{cases} \quad (16)$$

$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} *and* $\xi_{2t(l)}$

$$\vdots \qquad \vdots \qquad \vdots$$

The Autocorrelation of Z_{mt} and Z_{mt-l}

$$\rho_{mt,mt(l)} = \frac{\xi_{mt,mt(l)}}{\xi_{mt,mt}},$$

Where $\xi_{mt,mt}$ is given as $\sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt,nt(k)} + \sigma_{w_{mt}}^2$,   Usoro and Udoh (2023)

Therefore, from Equation (14)

$$\rho_{mt,mt(l)} = \begin{cases} 1 & ,l = 0 \\ \dfrac{\sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt(l-k),nt}}{\sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt,nt(k)} + \sigma_{w_{mt}}^2} & ,l = m \end{cases} \tag{17}$$

$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

Lower Diagonal VAR Models and Their Variances

This section considers the conditions for identification of the upper diagonal VAR models from the general form.

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & 0 & \cdots & 0 \\ \emptyset_{1.21} & \emptyset_{1.22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{1.m1} & \emptyset_{1.m2} & \cdots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix}$$
$$+ \begin{pmatrix} \emptyset_{2.11} & 0 & \cdots & 0 \\ \emptyset_{2.21} & \emptyset_{2.22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{2.m1} & \emptyset_{2.m2} & \cdots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix}$$
$$+ \ldots + \begin{pmatrix} \emptyset_{p.11} & 0 & \cdots & 0 \\ \emptyset_{p.21} & \emptyset_{p.22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{p.m1} & \emptyset_{p.m2} & \cdots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} +$$
$$\begin{pmatrix} w_{1t} \\ w_{2t} \\ \vdots \\ w_{mt} \end{pmatrix} \tag{18}$$

Usoro and Udoh (2023) presented the above set of models in the form,

$$Z_{it} = \begin{cases} \delta_1 + \varphi_{k.1j}Z_{jt-k} + w_{1t}, i = 1; j = 1; k = 1, \dots, p; (s \neq 1) \\ \delta_2 + \varphi_{k.2j}Z_{jt-k} + w_{2t}, i = 2; j = 1, 2; k = 1, \dots, p; (s \neq 2) \\ \delta_3 + \varphi_{k.3j}Z_{jt-k} + w_{3t}, i = 3; j = 1, 2, 3; k = 1, \dots, p; (s \neq 3) \\ \vdots \\ \delta_m + \varphi_{k.mj}Z_{jt-k} + w_{mt}, \ i = m; \ j = 1, 2, 3, \dots, n; \ k = 1, \dots, p; (s \neq m) \end{cases} \quad (19)$$

Equation (19) defines a set of Lower Diagonal Vector Autoregressive Models.

Autocovariances of Lower Diagonal VAR Models

Autocovariances of $Z_{1t}Z_{1t-l}$

$for\ \delta_1 = 0, i = 1$. multiply Equation (19) by Z_{1t-l} and take the expectations.

$$E(Z_{1t}Z_{1t-l}) = E[Z_{1t-l}(\emptyset_{1.11}Z_{1t-1} + \emptyset_{2.11}Z_{1t-2} + \dots + \emptyset_{p.11}Z_{1t-p} + \varepsilon_{1t})]$$

$$E(Z_{1t}Z_{1t-l}) = \emptyset_{1.11}E(Z_{1t-l}Z_{1t-1}) + \emptyset_{2.11}E(Z_{1t-l}Z_{1t-2}) + \dots + \emptyset_{p.11}E(Z_{1t-l}Z_{1t-p}) + E(Z_{1t-l}\varepsilon_{1t})$$

$$E(Z_{1t}Z_{1t-l}) = \xi_{1t,1t(l)}$$

$$\xi_{1t,1t(l)} = \emptyset_{1.11}\xi_{1t(l-1),1t} + \emptyset_{2.11}\xi_{1t(l-2),1t} + \dots + \emptyset_{p.11}\xi_{1t(l-p),1t}$$

where, $E(Z_{1t-l}\varepsilon_{1t}) = 0$ (uncorrelated stationary process)

$$\xi_{1t,1t(l)} = \sum_{k=1}^{p} \emptyset_{k.11}\xi_{1t(l-k),1t}, l = 1,2,3, \dots \quad (20)$$

Autocovariances of $Z_{2t}Z_{2t-l}$

$for\ \delta_2 = 0, i = 2$ multiply Equation (19) by Z_{2t-l} and take the expectations.

$$E(Z_{2t}Z_{2t-l}) = E[Z_{2t-l}(\emptyset_{1.21}Z_{1t-1} + \emptyset_{1.22}Z_{2t-1} + \emptyset_{2.21}Z_{1t-2} + \emptyset_{2.22}Z_{2t-2} + \dots + \emptyset_{p.21}Z_{1t-p} + \emptyset_{p.22}Z_{2t-p} + \varepsilon_{2t})]$$

$$E(Z_{2t}Z_{2t-l}) = \emptyset_{1.21}E(Z_{2t-l}Z_{1t-1}) + \emptyset_{1.22}E(Z_{2t-l}Z_{2t-1}) + \emptyset_{2.21}E(Z_{2t-l}Z_{1t-2}) + \emptyset_{2.22}E(Z_{2t-l}Z_{2t-2}) + \dots + \emptyset_{p.21}E(Z_{2t-l}Z_{1t-p}) + \emptyset_{p.22}E(Z_{2t-l}Z_{2t-p}) + E(Z_{2t-l}\varepsilon_{2t})$$

$$E(Z_{2t}Z_{2t-l}) = \xi_{2t,2t(l)}$$

$$\xi_{2t,2t(l)} = \emptyset_{1.21}\xi_{2t(l-1),1t} + \emptyset_{1.22}\xi_{2t(l-1)2t} + \dots + \emptyset_{2.21}\xi_{2t(l-2),1t} + \emptyset_{2.22}\xi_{2t(l-2),2t} + \dots + \emptyset_{p.21}\xi_{2t(l-p),1t} + \emptyset_{p.22}\xi_{2t(l-p),2t}$$

where, $E(Z_{2t-s}\varepsilon_{2t}) = 0$ (uncorrelated stationary process)

$$\xi_{2t,2t(l)} = \sum_{k=1}^{p}\sum_{j=1}^{2} \emptyset_{k.2j}\xi_{2t(l-k),jt}, l = 1,2,3, \dots \quad (21)$$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

Autocovariances $Z_{mt}Z_{mt-l}$

$for\ \delta_m = 0, l = m$. multiply Equation (19) by Z_{mt-l} and take the expectations.

$$E(Z_{mt}Z_{mt-l}) = E[Z_{mt-l}(\emptyset_{1.m1}Z_{1t-1} + \emptyset_{1.m2}Z_{2t-1} + \cdots + \emptyset_{1.mn}Z_{nt-1} + \emptyset_{2.m1}Z_{1t-2} + \emptyset_{2.m2}Z_{2t-2} + \cdots + \emptyset_{2.mn}Z_{nt-2} + \ldots + \emptyset_{p.m1}Z_{1t-p} + \emptyset_{p.m2}Z_{2t-p} + \cdots + \emptyset_{p.mn}Z_{nt-p} + \varepsilon_{mt})]$$

$$E(Z_{mt}Z_{mt-l}) = \emptyset_{1.m1}E(Z_{mt-l}Z_{1t-1}) + \emptyset_{1.m2}E(Z_{mt-l}Z_{2t-1}) + \cdots + \emptyset_{1.mn}E(Z_{mt-l}Z_{nt-1}) + \emptyset_{2.m1}E(Z_{mt-l}Z_{1t-2}) + \emptyset_{2.m2}E(Z_{mt-l}Z_{2t-2}) + \cdots + \emptyset_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \ldots + \emptyset_{p.m1}E(Z_{mt-l}Z_{1t-p}) + \emptyset_{p.m2}E(Z_{mt-l}Z_{2t-p}) + \cdots + \emptyset_{p.mn}E(Z_{mt-l}Z_{nt-p}) + E(Z_{mt-l}\varepsilon_{mt})$$

$$E(Z_{mt}Z_{mt-l}) = \xi_{mt,mt(l)}$$

$$\xi_{mt,mt(l)} = \emptyset_{1.m1}\xi_{mt(l-1),1t} + \emptyset_{1.m2}\xi_{mt,2t(l-1)} + \cdots + \emptyset_{1.mn}\xi_{mt,nt(l-1)} + \emptyset_{2.m1}\xi_{mt(l-2),1t} + \emptyset_{2.m2}\xi_{mt,2t(l-2)} + \cdots + \emptyset_{2.mn}\xi_{mt,nt(l-2)} + \cdots + \emptyset_{p.m1}\xi_{mt(l-p),1t} + \emptyset_{p.m2}\xi_{mt,2t(l-p)} + \cdots + \emptyset_{p.mn}\xi_{mt,nt(l-p)}$$

where, $E(Z_{mt-l}\varepsilon_{mt}) = 0$ (uncorrelated stationary process)

$$\xi_{mt,mt(l)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.mj}\xi_{mt(l-k),jt} \quad l = 1,2,3,\ldots \tag{22}$$

Autocorrelations of Lower Diagonal VAR Models

The Autocorrelation of Z_{1t} *and* Z_{1t-l}

$$\rho_{1t,1t(l)} = \frac{\xi_{1t,1t(l)}}{\xi_{1t,1t}},$$

Where $\xi_{1t,1t}$ is given as $\sum_{k=1}^{p} \emptyset_{k.11}\xi_{1t,1t(k)} + \sigma^2_{w_{1t}}$; Usoro and Udoh (2023)

Therefore, from Equation (20)

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\xi_{1t(l-k),jt}}{\sum_{k=1}^{p} \emptyset_{k.11}\xi_{1t,1t(k)} + \sigma^2_{w_{1t}}} & , l = 1,2,3,\ldots \end{cases} \tag{23}$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} *and* $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} *and* Z_{2t-l}

$\rho_{2t,2t(l)} = \frac{\xi_{2t,2t(l)}}{\xi_{2t,2t}}$,

Where $\xi_{2t,2t}$ is given as $\sum_{k=1}^{p}\sum_{j=1}^{2} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma^2_{w_{2t}}$; Usoro and Udoh (2023)

Therefore, from Equation (21)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & , l = 0 \\ \frac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\xi_{2t(l-k),jt}}{\sum_{k=1}^{p}\sum_{j=2}^{n} \emptyset_{k.2j}\xi_{2t,jt(k)} + \sigma^2_{w_{2t}}} & , l = 1,2,3,\dots \end{cases} \quad (24)$$

$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} *and* $\xi_{2t(l)}$

⋮ ⋮ ⋮

The Autocorrelation of Z_{mt} and Z_{mt-l}

$\rho_{mt,mt(l)} = \frac{\xi_{mt,mt(l)}}{\xi_{mt,mt}}$,

Where $\xi_{mt,mt}$ is given as $\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.mj}\xi_{mt,jt(k)} + \sigma^2_{w_{mt}}$, Usoro and Udoh (2023)

Therefore, from Equation (22)

$$\rho_{mt,mt(l)} = \begin{cases} 1 & , l = 0 \\ \frac{\sum_{k=1}^{p} \emptyset_{k.mn}\xi_{mt(l-k),nt}}{\sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.mj}\xi_{mt,jt(k)} + \sigma^2_{w_{mt}}} & , l = 1,2,3,\dots \end{cases} \quad (25)$$

$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

General MARDL Model.
Definition

Let $\underline{Z}_t = (Z_{1t}, Z_{2t}, \dots, Z_{mt})^I$ be the vector of response time variables, $\underline{\emptyset} = (\emptyset_{k.ij})$ is the coefficients vector, $\underline{Z}_{st}(s \neq i)$ is the non-lag predictor, $\underline{\emptyset}_{st(s\neq i)}$ is a vector of non-lag coefficients, $\underline{Z}_{t-k} = (Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})^I$ is defined as the vector of the predictive lag time variables, $\underline{\delta} = (\delta_1, \delta_2, \dots, \delta_m)^I$ is the vector of constants and $\underline{\varepsilon}_t = (\varepsilon_{t1}, \varepsilon_{2t}, \dots, \varepsilon_{mt})^I$ is the vector of error terms associated with the vector of response time variables.

The above definition is presented in the following form,

$$(Z_{1t}, Z_{2t}, \dots, Z_{mt})^I = (\delta_1, \delta_2, \dots, \delta_m)^I + (\emptyset_{is})(Z_{1t}, Z_{2t}, \dots, Z_{nt})^I + (\emptyset_{k.ij})(Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})^I + (\varepsilon_{t1}, \varepsilon_{2t}, \dots, \varepsilon_{mt})^I \quad (26)$$

The expanded form of Equation (26) is

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} 0 & \emptyset_{12} & \cdots & \emptyset_{1n} \\ \emptyset_{21} & 0 & \dots & \emptyset_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{m1} & \emptyset_{m2} & \dots & 0 \end{pmatrix} \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & \emptyset_{1.12} & \cdots & \emptyset_{1.1n} \\ \emptyset_{1.21} & \emptyset_{1.22} & \dots & \emptyset_{1.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{1.m1} & \emptyset_{1.m2} & \dots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix} + \begin{pmatrix} \emptyset_{2.11} & \emptyset_{2.12} & \cdots & \emptyset_{2.1n} \\ \emptyset_{2.21} & \emptyset_{2.22} & \dots & \emptyset_{2.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{2.m1} & \emptyset_{2.m2} & \dots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix} + \dots + \begin{pmatrix} \emptyset_{p.11} & \emptyset_{p.12} & \cdots & \emptyset_{p.1n} \\ \emptyset_{p.21} & \emptyset_{p.22} & \dots & \emptyset_{p.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{p.m1} & \emptyset_{p.m2} & \dots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{mt} \end{pmatrix} \quad (27)$$

Equation (27) is reduced to the form,

$$Z_{it} = \delta_i + \sum_{s=1}^{m} \emptyset_{is} Z_{st} + \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.ij} Z_{jt-k} + \varepsilon_{jt}, i = 1, \dots, m (m = n), (i \neq s) \quad (28)$$

$\emptyset_{is}$ is a matrix non-lag coefficient of the predictor variables, $\emptyset_{k.ij}$ are matrices of coefficients of j predictors to i responses at k lags, $\delta_{i(i=1,\dots,m)}$ are constants.

The set of General Multivariate Autoregressive Distributed Lag Model Is defined below,

$$Z_{it} = \begin{cases} \delta_1 + \emptyset_{1s} Z_{st} + \emptyset_{k.1j} Z_{jt-k} + \varepsilon_{1t}, i = 1; j = 1, \dots, n; k = 1, \dots, p; (s \neq 1) \\ \delta_2 + \emptyset_{2s} Z_{st} + \emptyset_{k.2j} Z_{jt-k} + \varepsilon_{2t}, i = 2; j = 1, \dots, n; k = 1, \dots, p; (s \neq 2) \\ \delta_3 + \emptyset_{3s} Z_{st} + \emptyset_{k.3j} Z_{jt-k} + \varepsilon_{3t}, i = 3; j = 1, \dots, n; k = 1, \dots, p; (s \neq 3) \\ \vdots \\ \delta_m + \emptyset_{ms} Z_{st} + \emptyset_{k.mj} Z_{jt-k} + \varepsilon_{mt}, \; i = m; \; j = n; \quad k = 1, \dots, p; (s \neq m) \end{cases} \quad (29)$$

Autocovariances of the General MARDL Models.

Autocovariances of $Z_{1t}Z_{1t-l}$

$for\ \delta_1 = 0, i = 1$, Multiplying Equation (29) by Z_{1t-l} and taking the expectations,

$$E(Z_{1t}Z_{1t-l}) = E\big[Z_{1t-l}\big(\varphi_{12}Z_{2t} + \varphi_{13}Z_{3t} + \dots + \varphi_{1n}Z_{nt} + \varphi_{1.11}Z_{1t-1} + \varphi_{1.12}Z_{2t-1} + \cdots + \varphi_{1.1n}Z_{nt-1} + \varphi_{2.11}Z_{1t-2} + \varphi_{2.12}Z_{2t-2} + \cdots + \varphi_{2.1n}Z_{nt-2} + \cdots + \varphi_{p.11}Z_{1t-p} + \varphi_{p.12}Z_{2t-p} + \cdots + \varphi_{p.1n}Z_{nt-p} + \varepsilon_{1t}\big)\big]$$

$$\begin{aligned} E(Z_{1t}Z_{1t-l}) = &\varphi_{12}E(Z_{1t-l}Z_{2t}) + \varphi_{13}E(Z_{1t-l}Z_{3t}) + \cdots + \varphi_{1n}E(Z_{1t-l}Z_{nt}) \\ &+ \varphi_{1.11}E(Z_{1t-l}Z_{1t-1}) + \varphi_{1.12}E(Z_{1t-l}Z_{2t-1}) + \cdots + \varphi_{1.1n}E(Z_{1t-l}Z_{nt-1}) \\ &+ \varphi_{2.11}E(Z_{1t-l}Z_{1t-2}) + \varphi_{2.12}E(Z_{1t-l}Z_{2t-2}) \ldots + \varphi_{2.1n}E(Z_{1t-l}Z_{nt-2}) + \cdots \\ &+ \varphi_{p.11}E(Z_{1t-l}Z_{1t-p}) + \varphi_{p.12}E(Z_{1t-l}Z_{2t-p} + \cdots + \varphi_{p.1n}E(Z_{1t-l}Z_{nt-p}) \\ &+ E(Z_{1t-l}\varepsilon_{1t}) \end{aligned}$$

$$\begin{aligned} \xi_{1t,1t(l)} = &\varphi_{12}\xi_{1t(l),2t} + \varphi_{13}\xi_{1t(l),3t} + \cdots + \varphi_{1n}\xi_{1t(l),nt} + \varphi_{1.11}\xi_{1t(l-1),1t} + \varphi_{1.12}\xi_{1t(l-1),2t} + \cdots \\ &+ \varphi_{1.1n}\xi_{1t(l-1),nt} + \varphi`_{2.11}\xi_{1t,(l-2)1t} + \varphi`_{2.12}\xi_{1t(l-2),2t} + \cdots + \varphi_{2.1n}\xi_{1t,(l-2)nt} \\ &+ \cdots + \varphi_{p.11}\xi_{1t,1t(l-p)} + \varphi_{p.12}\xi_{1t,2t(l-p)} \ldots + \varphi_{p.1n}\xi_{1t,nt(l-p)} \end{aligned}$$

$E(\xi_{1t-l}\varepsilon_{1t}) = 0$ (uncorrelated processes)

$$\xi_{1t,\,1t(l)} = \sum_{s=2}^{m} \varphi_{1s}\ \xi_{1t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\,\xi_{1t,\ jt(l-k)}\,, (s \neq 1),\ l = 1,2,3,\ldots \qquad (30)$$

Autocovariance of $Z_{2t}Z_{2t-l}$

$for\ \delta_2 = 0, i = 2$, Multiplying Equation (29) by Z_{2t-l} and taking the expectations,

$$\begin{aligned} E(Z_{2t}Z_{2t-l}) = \ &E[Z_{2t-l}(\emptyset_{21}Z_{1t} + \emptyset_{23}Z_{3t} + \ldots + \emptyset_{2n}Z_{nt} + \emptyset_{1.21}Z_{1t-1} + \emptyset_{1.22}Z_{2t-1} + \cdots \\ &+ \emptyset_{1.2n}Y_{nt-1} + \emptyset_{2.21}Z_{1t-2} + \emptyset_{2.22}Z_{2t-2} + \cdots + \emptyset_{2.2n}Z_{nt-2} + \cdots + \emptyset_{p.21}Z_{1t-p} \\ &+ \emptyset_{p.22}\emptyset_{2t-p} + \cdots + \emptyset_{p.2n}Z_{nt-p} + \varepsilon_{2t})] \end{aligned}$$

$$\begin{aligned} E(Z_{2t}Z_{2t-l}) = &\emptyset_{21}E(Z_{2t-l}Z_{1t}) + \emptyset_{23}E(Z_{2t-l}Z_{3t}) + \cdots + \emptyset_{2n}E(Z_{2t-l}Z_{nt}) \\ &+ \emptyset_{1.21}E(Z_{2t-l}Z_{1t-1}) + \emptyset_{1.22}E(Z_{2t-l}Z_{2t-1}) + \cdots + \emptyset_{1.2n}E(Z_{2t-l}Z_{nt-1}) \\ &+ \emptyset`_{2.21}E(Z_{2t-l}Z_{1t-2}) + \emptyset`_{2.22}E(Z_{2t-l}Z_{2t-2}) + \cdots + \emptyset_{2.2n}E(Z_{2t-l}Z_{nt-2}) + \cdots \\ &+ \emptyset_{p.21}E(Z_{2t-l}Z_{1t-p}) + \emptyset_{p.22}E(Z_{2t-l}Z_{2t-p}) \ldots + \emptyset_{p.2n}E(Z_{2t-l}Z_{nt-p}) \\ &+ E(Z_{2t-l}\varepsilon_{1t}) \end{aligned}$$

$$\begin{aligned} \xi_{2t,2t(l)} = &\emptyset_{21}\xi_{2t(l),1t} + \emptyset_{23}\xi_{2t(l),3t} + \cdots + \emptyset_{2n}\xi_{2t(l),nt} + \emptyset_{1.21}\xi_{2t(l-1),1t} + \emptyset_{1.22}\xi_{2t(l-1),2t} + \cdots \\ &+ \emptyset_{1.2n}\xi_{2t(l-1),nt} + \emptyset_{2.21}\xi_{2t(l-2),1t} + \emptyset_{2.22}\xi_{2t(l-2),2t} + \cdots + \emptyset_{2.2n}\xi_{2t(l-2),nt} + \cdots \\ &+ \emptyset_{p.21}\xi_{2t(l-p),1t} + \emptyset_{p.22}\xi_{2t(l-p),2t} + \cdots + \emptyset_{p.2n}\xi_{2t(l-p),nt} \end{aligned}$$

$E(\xi_{2t-l}\varepsilon_{2t}) = 0$ (uncorrelated processes)

$$\xi_{2t,\,2t(l)} = \sum_{s=2}^{m} \varphi_{2s}\ \xi_{2t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\,\xi_{2t,\ jt(l-k)}\,, (s \neq 2),\ l = 1,2,3,\ldots \qquad (31)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Autocovariance of $Z_{mt}Z_{mt-l}$

$for\ \delta_m = 0, i = m$, Multiplying Equation (29) by Z_{mt-l} and taking the expectations,

$$\begin{aligned} E(Z_{mt}Z_{mt-l}) = \ &E[Z_{mt-l}(\emptyset_{m1}Z_{1t} + \emptyset_{m2}Z_{2t} + \ldots + \emptyset_{mn}Z_{nt} + \emptyset_{1.m1}Z_{1t-1} + \emptyset_{1.m2}Z_{2t-1} + \cdots \\ &+ \emptyset_{1.mn}Y_{nt-1} + \emptyset_{2.m1}Z_{1t-2} + \emptyset_{2.m2}Z_{2t-2} + \cdots + \emptyset_{2.mn}Z_{nt-2} + \cdots \\ &+ \emptyset_{p.m1}Z_{1t-p} + \emptyset_{p.m2}\emptyset_{2t-p} + \cdots + \emptyset_{p.mn}Z_{nt-p} \\ &+ \varepsilon_{mt})] \end{aligned}$$

$$E(Z_{mt}Z_{mt-l}) = \emptyset_{m1}E(Z_{mt-l}Z_{1t}) + \emptyset_{m2}E(Z_{mt-l}Z_{2t}) + \cdots + \emptyset_{mn}E(Z_{mt-l}Z_{nt}) + \emptyset_{1.m1}E(Z_{mt-l}Z_{1t-1}) + \emptyset_{1.m2}E(Z_{mt-l}Z_{2t-1}) + \cdots + \emptyset_{1.mn}E(Z_{mt-l}Z_{nt-1}) + \emptyset`_{2.m1}E(Z_{mt-l}Z_{1t-2}) + \emptyset`_{2.m2}E(Z_{mt-l}Z_{2t-2}) + \cdots + \emptyset_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \cdots + \emptyset_{p.m1}E(Z_{mt-l}Z_{1t-p}) + \emptyset_{p.m2}E(Z_{mt-l}Z_{2t-p}) \ldots + \emptyset_{p.mn}E(Z_{mt-l}Z_{nt-p}) + E(Z_{mt-l}\varepsilon_{mt})$$

$$\xi_{mt,mt(l)} = \emptyset_{m1}\xi_{mt(l),1t} + \emptyset_{m2}\xi_{mt(l),2t} + \cdots + \emptyset_{mn}\xi_{mt(l),nt} + \emptyset_{1.m1}\xi_{mt(l-1),1t} + \emptyset_{1.m2}\xi_{mt(l-1),2t} + \cdots + \emptyset_{1.mn}\xi_{mt(l-1),nt} + \emptyset_{2.m1}\xi_{mt(l-2),1t} + \emptyset_{2.m2}\xi_{mt(l-2),2t} + \cdots + \emptyset_{2.mn}\xi_{mt(l-2),nt} + \cdots + \emptyset_{p.m1}\xi_{mt(l-p),1t} + \emptyset_{p.m2}\xi_{mt(l-p),2t} + \cdots + \emptyset_{p.mn}\xi_{mt(l-p),nt}$$

$E(\xi_{mt-l}\varepsilon_{mt}) = 0$ (uncorrelated processes)

$$\xi_{mt,mt(l)} = \sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{mt(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(l-k)}, \quad s \neq m, l = 1,2,3, \ldots \qquad (32)$$

Autocorrelations of General MARDL Models

The Autocorrelation of Z_{1t} and Z_{1t-l}

$$\rho_{1t,1t(l)} = \frac{\xi_{1t,1t(l)}}{\xi_{1t,1t}},$$

Where $\xi_{1t,1t}$ is given as $\sum_{s=2}^{m} \varphi_{1s}\, \xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\, \xi_{1t,jt(k)} + \sigma^2_{w_{1t}}$, $s \neq 1$; Usoro and Udoh (2023)

Therefore, from Equation (30)

$$\rho_{1t,1t(l)} = \begin{cases} 1, & l = 0 \\ \dfrac{\sum_{s=2}^{m} \varphi_{1s}\ \xi_{1t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\, \xi_{1t,\ jt(l-k)}}{\sum_{s=2}^{m} \varphi_{1s}\, \xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\, \xi_{1t,jt(k)} + \sigma^2_{w_{1t}}}, & (s \neq 1),\ l = 1,2,3, \ldots \end{cases} \qquad (33)$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} and $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} and Z_{2t-l}

$$\rho_{2t,2t(l)} = \frac{\xi_{2t,2t(l)}}{\xi_{2t,2t}},$$

Where $\xi_{2t,2t}$ is given as $\sum_{s=1}^{m} \varphi_{2s}\, \xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,jt(k)} + \sigma^2_{w_{2t}}$, $s \neq 2$; Usoro and Udoh (2023)

Therefore, from Equation (31)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & ,l=0 \\ \dfrac{\sum_{s=2}^{m} \varphi_{2s}\ \xi_{2t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,\ jt(l-k)}}{\sum_{s=1}^{m} \varphi_{2s}\, \xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,jt(k)} + \sigma_{w_{2t}}^2} & ,(s \neq 2),\ l = 1,2,3,\dots \end{cases} \quad (34)$$

$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} *and* $\xi_{2t(l)}$

$\vdots \qquad \vdots \qquad \vdots$

The Autocorrelation of Z_{mt} and Z_{mt-l}

$\rho_{mt,mt(l)} = \dfrac{\xi_{mt,mt(l)}}{\xi_{mt,mt}}$,

Where $\xi_{mt,mt}$ is given as $\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{ms} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(k)} + \sigma_{w_{mt}}^2,\quad s \neq m$; Usoro and Udoh (2023)

Therefore, from Equation (32)

$$\rho_{mt,mt(l)} = \begin{cases} 1 & ,l=0 \\ \dfrac{\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{mt(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(l-k)}}{\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{ms} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(k)} + \sigma_{w_{mt}}^2} & ,s \neq m, l = 1,2,3,\dots \end{cases} \quad (35)$$

$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

Upper Diagonal MARDL Model

Model Derivation

This section considers the conditions for identification of the upper diagonal MARDL models from the general form.

From Equation (27), the following set of models is obtained

$$Z_{it} = \begin{cases} \delta_1 + \emptyset_{1s} Z_{st} + \emptyset_{k.1j} Z_{jt-k} + \varepsilon_{1t}, i = 1; j = 1, \dots, n; k = 1, \dots, p; (s \neq 1) \\ \delta_2 + \emptyset_{2s} Z_{st} + \emptyset_{k.2j} Z_{jt-k} + \varepsilon_{2t}, i = 2; j = 2, \dots, n; k = 1, \dots, p; (s \neq 2) \\ \delta_3 + \emptyset_{3s} Z_{st} + \emptyset_{k.3j} Z_{jt-k} + \varepsilon_{3t}, i = 3; j = 3, \dots, n; k = 1, \dots, p; (s \neq 3) \\ \vdots \\ \delta_m + \emptyset_{ms} Z_{st} + \emptyset_{k.mj} Z_{jt-k} + \varepsilon_{mt},\ i = m;\ j = n;\ \ k = 1, \dots, p; (s \neq m) \end{cases} \quad (36)$$

Equation (36) defines a set of Upper Diagonal Multivariate Autoregressive Distributed Lag Models (UDMARDL).

Model Expansion:

Given Equation (28)

$$Z_{it} = \delta_i + \sum_{s=1}^{m} \emptyset_{is} Z_{st} + \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.ij}\, Z_{jt-k} + \varepsilon_{jt}, i = 1, \dots, m\ (i \neq s)$$

Autocovariance of Upper Diagonal MARDL Model

This section considers derivation of autocovariances of Upper Diagonal Multivariate Autoregressive Distributed Lag (MARDL) Models,

Autocovariance of $Z_{1t}Z_{1t-l}$

$for\ \delta_1 = 0, i = 1$, Multiplying Equation (36) by Z_{1t-l} and taking the expectations,

$$E(Z_{1t}Z_{1t-l}) = E\big[Z_{1t-l}\big(\varphi_{12}Z_{2t} + \varphi_{13}Z_{3t} + \dots + \varphi_{1n}Z_{nt} + \varphi_{1.11}Z_{1t-1} + \varphi_{1.12}Z_{2t-1} + \cdots + \varphi_{1.1n}Z_{nt-1} + \varphi_{2.11}Z_{1t-2} + \varphi_{2.12}Z_{2t-2} + \cdots + \varphi_{2.1n}Z_{nt-2} + \cdots + \varphi_{p.11}Z_{1t-p} + \varphi_{p.12}Z_{2t-p} + \cdots + \varphi_{p.1n}Z_{nt-p} + \varepsilon_{1t}\big)\big]$$

$$E(Z_{1t}Z_{1t-l}) = \varphi_{12}E(Z_{1t-l}Z_{2t}) + \varphi_{13}E(Z_{1t-l}Z_{3t}) + \cdots + \varphi_{1n}E(Z_{1t-l}Z_{nt}) + \varphi_{1.11}E(Z_{1t-l}Z_{1t-1}) + \varphi_{1.12}E(Z_{1t-l}Z_{2t-1}) + \cdots + \varphi_{1.1n}E(Z_{1t-l}Z_{nt-1}) + \varphi_{2.11}E(Z_{1t-l}Z_{1t-2}) + \varphi_{2.12}E(Z_{1t-l}Z_{2t-2}) \dots + \varphi_{2.1n}E(Z_{1t-l}Z_{nt-2}) + \cdots + \varphi_{p.11}E(Z_{1t-l}Z_{1t-p}) + \varphi_{p.12}E(Z_{1t-l}Z_{2t-p} + \cdots + \varphi_{p.1n}E\big(Z_{1t-l}Z_{nt-p}\big) + E(Z_{1t-l}\varepsilon_{1t})$$

$$\xi_{1t,1t(l)} = \varphi_{12}\xi_{1t(l),2t} + \varphi_{13}\xi_{1t(l),3t} + \cdots + \varphi_{1n}\xi_{1t(l),nt} + \varphi_{1.11}\xi_{1t(l-1),1t} + \varphi_{1.12}\xi_{1t(l-1),2t} + \cdots + \varphi_{1.1n}\xi_{1t(l-1),nt} + \varphi`_{2.11}\xi_{1t,(l-2)1t} + \varphi`_{2.12}\xi_{1t(l-2),2t} + \cdots + \varphi_{2.1n}\xi_{1t,(l-2)nt} + \cdots + \varphi_{p.11}\xi_{1t,1t(l-p)} + \varphi_{p.12}\xi_{1t,2t(l-p)} \dots + \varphi_{p.1n}\xi_{1t,nt(l-p)}$$

$E(\xi_{1t-l}\varepsilon_{1t}) = 0$ (uncorrelated processes)

$$\xi_{1t,\,1t(l)} = \sum_{s=2}^{m} \varphi_{1s}\ \xi_{1t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\,\xi_{1t,\ jt(L-k)}, (s \neq 1),\ l = 1,2,3, \dots \qquad (37)$$

Autocovariance of $Z_{2t}Z_{2t-l}$

$for\ 2 = 0, i = 2$, Multiplying Equation (36) by Z_{2t-l} and taking the expectations,

$$E(Z_{2t}Z_{2t-l}) = \varphi_{21}E(Z_{2t-l}Z_{1t}) + \varphi_{23}E(Z_{2t-l}Z_{3t}) + \cdots + \varphi_{2n}E(Z_{2t-l}Z_{nt}) + \varphi_{1.21}E(Z_{2t-l}Z_{1t-1}) + \varphi_{1.22}E(Z_{2t-l}Z_{2t-1}) + \cdots + \varphi_{1.2n}E(Z_{2t-l}Z_{nt-1}) + \varphi_{2.21}E(Z_{2t-l}Z_{1t-2}) + \varphi_{2.22}E(Z_{2t-l}Z_{2t-2}) \dots + \varphi_{2.2n}E(Z_{2t-l}Z_{nt-2}) + \cdots + \varphi_{p.21}E(Z_{2t-l}Z_{1t-p}) + \varphi_{p.22}E(Z_{2t-l}Z_{2t-p} + \cdots + \varphi_{p.2n}E\big(Z_{2t-l}Z_{nt-p}\big) + E(Z_{2t-l}\varepsilon_{1t})$$

$$\xi_{2t,2t(l)} = \varphi_{21}\xi_{2t(l),1t} + \varphi_{23}\xi_{2t(l),3t} + \cdots + \varphi_{2n}\xi_{2t(l),nt} + \varphi_{1.21}\xi_{2t(l-1),1t} + \varphi_{1.22}\xi_{2t(l-1),2t} + \cdots + \varphi_{1.2n}\xi_{2t(l-1),nt} + \varphi`_{2.21}\xi_{2t,(l-2)1t} + \varphi`_{2.22}\xi_{2t(l-2),2t} + \cdots + \varphi_{2.2n}\xi_{2t,(l-2)nt} + \cdots + \varphi_{p.21}\xi_{2t,1t(l-p)} + \varphi_{p.22}\xi_{2t,2t(l-p)} \ldots + \varphi_{p.2n}\xi_{2t,nt(l-p)}$$

$E(\xi_{2t-l}\varepsilon_{1t}) = 0$ (uncorrelated processes)

$$\xi_{2t,2t(l)} = \sum_{s=3}^{m} \varphi_{2s}\ \xi_{2t\,(l),st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\,\xi_{2t,jt(l-k)}, \qquad (s \neq 2), l = 1,2,3, \ldots \qquad (38)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Autocovariance of $Z_{mt}Z_{mt-l}$

$for\ \delta_m = 0, i = 1$, Multiplying Equation (36) by Z_{mt-l} and taking the expectations,

$$E(Z_{mt}Z_{mt-l}) = E[Z_{mt-l}(\emptyset_{m1}Z_{1t} + \emptyset_{m2}Z_{2t} + \ldots + \emptyset_{mn}Z_{nt} + \emptyset_{1.m1}Z_{1t-1} + \emptyset_{1.m2}Z_{2t-1} + \cdots + \emptyset_{1.mn}Y_{nt-1} + \emptyset_{2.m1}Z_{1t-2} + \emptyset_{2.m2}Z_{2t-2} + \cdots + \emptyset_{2.mn}Z_{nt-2} + \cdots + \emptyset_{p.m1}Z_{1t-p} + \emptyset_{p.m2}\emptyset_{2t-p} + \cdots + \emptyset_{p.mn}Z_{nt-p} + \varepsilon_{mt})]$$

$$E(Z_{mt}Z_{mt-l}) = \varphi_{m1}E(Z_{mt-l}Z_{1t}) + \varphi_{m2}E(Z_{mt-l}Z_{2t}) + \cdots + \varphi_{mn}E(Z_{mt-l}Z_{nt}) + \varphi_{1.m1}E(Z_{mt-l}Z_{1t-1}) + \varphi_{1.m2}E(Z_{mt-l}Z_{2t-1}) + \cdots + \varphi_{1.mn}E(Z_{mt-l}Z_{nt-1}) + \varphi_{2.m1}E(Z_{mt-l}Z_{1t-2}) + \varphi_{2.m2}E(Z_{mt-l}Z_{2t-2}) \ldots + \varphi_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \cdots + \varphi_{p.m1}E(Z_{mt-l}Z_{1t-p}) + \varphi_{p.m2}E(Z_{mt-l}Z_{2t-p} + \cdots + \varphi_{p.mn}E(Z_{mt-l}Z_{nt-p}) + E(Z_{mt-l}\varepsilon_{1t})$$

$$\xi_{mt,mt(l)} = \varphi_{m1}\xi_{mt(l),1t} + \varphi_{m2}\xi_{mt(l),3t} + \cdots + \varphi_{mn}\xi_{mt(l),nt} + \varphi_{1.m1}\xi_{mt(l-1),1t} + \varphi_{1.m2}\xi_{mt(l-1),2t} + \cdots + \varphi_{1.mn}\xi_{mt(l-1),nt} + \varphi`_{2.m1}\xi_{mt,(l-2)1t} + \varphi`_{2.m2}\xi_{mt(l-2),2t} + \cdots + \varphi_{2.mn}\xi_{mt,(l-2)nt} + \cdots + \varphi_{p.m1}\xi_{mt,1t(l-p)} + \varphi_{p.m2}\xi_{mt,2t(l-p)} \ldots + \varphi_{p.mn}\xi_{mt,nt(l-p)},$$

$E(\xi_{mt-l}\varepsilon_{mt}) = 0$ (uncorrelated processes)

$$\xi_{mt,mt(l)} = \sum_{s=1}^{m-1} \varphi_{ms}\xi_{mt\,(l),\ st} + \sum_{J=1}^{p}\sum_{k=1}^{n} \varphi_{k.mj}\,\xi_{mt,\ jt(l-k)}, \quad (s \neq m), l = 1,2,3, \ldots \qquad (39)$$

Autocorrelation of Upper Diagonal MARDL Models

The Autocorrelation of Z_{1t} and Z_{1t-l}

$$\rho_{1t,1t(l)} = \frac{\xi_{1t,1t(l)}}{\xi_{1t,1t}},$$

Where $\xi_{1t,1t}$ is given as $\sum_{s=2}^{m} \varphi_{1s}\,\xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\,\xi_{1t,jt(k)} + \sigma^2_{w_{1t}},\ s \neq 1;$ Usoro and Udoh (2023)

Therefore, from Equation (37)

$$\rho_{1t,1t(l)} = \begin{cases} 1 & ,l = 0 \\ \dfrac{\sum_{s=2}^{m} \varphi_{1s}\ \xi_{1t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\, \xi_{1t,\ jt(l-k)}}{\sum_{s=2}^{m} \varphi_{1s}\, \xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\, \xi_{1t,jt(k)} + \sigma_{w_{1t}}^2} & ,(s \neq 1),\ l = 1,2,3,\ldots \end{cases} \quad (40)$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} *and* $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} and Z_{2t-l}

$\rho_{2t,2t(l)} = \dfrac{\xi_{2t,2t(l)}}{\xi_{2t,2t}}$,

Where $\xi_{2t,2t}$ is given as $\sum_{s=1}^{m} \varphi_{2s}\, \xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,jt(k)} + \sigma_{w_{2t}}^2$, $s \neq 2$; Usoro and Udoh (2023)

Therefore, from Equation (38)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & ,l = 0 \\ \dfrac{\sum_{s=2}^{m} \varphi_{2s}\ \xi_{2t\,(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,\ jt(l-k)}}{\sum_{s=1}^{m} \varphi_{2s}\, \xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\, \xi_{2t,jt(k)} + \sigma_{w_{2t}}^2} & ,(s \neq 2),\ l = 1,2,3,\ldots \end{cases} \quad (41)$$

$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} *and* $\xi_{2t(l)}$

⋮ ⋮ ⋮

The Autocorrelation of Z_{mt} and Z_{mt-l}

$\rho_{mt,mt(l)} = \dfrac{\xi_{mt,mt(l)}}{\xi_{mt,mt}}$,

Where $\xi_{mt,mt}$ is given as $\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{ms} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(k)} + \sigma_{w_{mt}}^2$, $s \neq m$; Usoro and Udoh (2023)

Therefore, from Equation (39)

$$\rho_{mt,mt(l)} = \begin{cases} 1 & ,l=0 \\ \dfrac{\sum_{s=1}^{m-1}\varphi_{ms}\,\xi_{mt(l),\ st} + \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\,\xi_{mt,jt(l-k)}}{\sum_{s=1}^{m-1}\varphi_{ms}\,\xi_{ms} + \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\,\xi_{mt,jt(k)} + \sigma_{w_{mt}}^{2}} & ,s \neq m,l \end{cases}$$
$$= 1,2,3,\ldots \qquad (42)$$

$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

Lower Diagonal MARDL Model

Model Derivation

This section considers the conditions for identification of the lower diagonal MARDL models from the general form. From Equation (27), we have a set the following lower diagonal models:

$$Z_{it} = \begin{cases} \delta_1 + \emptyset_{1s}Z_{st} + \emptyset_{k.1j}Z_{jt-k} + \varepsilon_{1t}, i=1; j=1; k=1,\ldots,p; (s\neq 1) \\ \delta_2 + \emptyset_{2s}Z_{st} + \emptyset_{k.2j}Z_{jt-k} + \varepsilon_{2t}, i=2; j=1,2; k=1,\ldots,p; (s\neq 2) \\ \delta_3 + \emptyset_{3s}Z_{st} + \emptyset_{k.3j}Z_{jt-k} + \varepsilon_{3t}, i=3; j=1,2,3; k=1,\ldots,p; (s\neq 3) \\ \vdots \\ \delta_m + \emptyset_{ms}Z_{st} + \emptyset_{k.mj}Z_{jt-k} + \varepsilon_{mt}, \ i=m;\ j=1,2,3,\ldots,n; \quad k=1,\ldots,p; (s\neq m) \end{cases} \qquad (43)$$

Equation (43) defines a set of Lower Diagonal Multivariate Autoregressive Distributed Lag Models (LDMARDL).

Model Expansion:

Given Equation (28) as

$$Z_{it} = \delta_i + \sum_{s=1}^{m}\emptyset_{is}Z_{st} + \sum_{k=1}^{p}\sum_{j=1}^{n}\emptyset_{k.ij}\,Z_{jt-k} + \varepsilon_{jt}, i=1,\ldots,m\ (i\neq s)$$

Autocovariance of Lower Diagonal MARDL Model

This section considers derivations of autocovariances of Lower Diagonal of Multivariate Autoregressive Distributed Lag (MARDL) Models

Autocovariance of $\boldsymbol{Z_{1t}Z_{1t-l}}$

$for\ \delta_1 = 0, i = 1,$ Multiplying Equation (43) by Z_{1t-l} and taking the expectations,

$$E(Z_{1t}Z_{1t-l}) = E\big[Z_{1t-l}\big(\emptyset_{12}Z_{2t} + \emptyset_{13}Z_{3t} + \ldots + \emptyset_{1n}Z_{nt} + \emptyset_{1.11}Z_{1t-1} + \emptyset_{2.11}Z_{1t-2} + \ \ldots \\ + \ \emptyset_{p.11}Z_{1t-p} + \varepsilon_{1t}\big)\big]$$

$$E(Z_{1t}Z_{1t-l}) = \varphi_{12}E(Z_{1t-l}Z_{2t}) + \varphi_{13}E(Z_{1t-l}Z_{3t}) + \cdots + \varphi_{1n}E(Z_{1t-l}Z_{nt}) \\ + \varphi_{1.11}E(Z_{1t-l}Z_{1t-1}) + \varphi_{2.11}E(Z_{1t-l}Z_{1t-2}) + \varphi_{p.11}E(Z_{1t-l}Z_{1t-p}) \\ + E(Z_{1t-l}\varepsilon_{1t})$$

$$\xi_{1t,1t(l)} = \varphi_{12}\xi_{1t(l),2t} + \varphi_{13}\xi_{1t(l),3t} + \cdots + \varphi_{1n}\xi_{1t(l),nt} + \varphi_{1.11}\xi_{1t(l-1),1t} + \varphi`_{2.11}\xi_{1t,(l-2)1t} \\ + \varphi_{p.11}\xi_{1t,1t(l-p)}$$

$E(\xi_{1t-l}\varepsilon_{1t}) = 0$ (uncorrelated processes)

$$\xi_{1t,1t(l)} = \sum_{k=1}^{p} \varphi_{k.11}\, \xi_{1t,kt(l-p)}, \tag{44}$$

Autocovariance of $Z_{2t}Z_{2t-l}$
$for\ \delta_2 = 0, i = 2$, Multiplying Equation (43) by Z_{2t-l} and taking the expectations,

$$E(Z_{2t}Z_{2t-l}) = E\big[Z_{2t-l}\big(\emptyset_{21}Z_{1t} + \emptyset_{23}Z_{3t} + \ldots + \emptyset_{2n}Z_{nt} + \emptyset_{1.21}Z_{1t-1} + \emptyset_{1.22}Z_{2t-1} + \emptyset_{2.21}Z_{1t-2} + \emptyset_{2.22}Z_{2t-2} + \cdots + \emptyset_{p.21}Z_{1t-p} + \emptyset_{p.22}Z_{2t-p} + \varepsilon_{2t}\big)\big]$$

$$E(Z_{2t}Z_{2t-l}) = \varphi_{21}E(Z_{2t-l}Z_{1t}) + \varphi_{23}E(Z_{2t-l}Z_{3t}) + \cdots + \varphi_{2n}E(Z_{2t-l}Z_{nt}) + \varphi_{1.21}E(Z_{2t-l}Z_{1t-1}) + \varphi_{1.22}E(Z_{2t-l}Z_{2t-1}) + \varphi_{2.21}E(Z_{2t-l}Z_{1t-2}) + \varphi_{2.22}E(Z_{2t-l}Z_{2t-2}) + \cdots + \varphi_{p.21}E(Z_{2t-l}Z_{1t-p}) + \varphi_{p.22}E(Z_{2t-l}Z_{2t-p} + E(Z_{2t-l}\varepsilon_{1t})$$

$$\xi_{2t,2t(l)} = \varphi_{21}\xi_{2t(l),1t} + \varphi_{23}\xi_{2t(l),3t} + \cdots + \varphi_{2n}\xi_{2t(l),nt} + \varphi_{1.21}\xi_{2t(l-1),1t} + \varphi_{1.22}\xi_{2t(l-1),2t} + \varphi`_{2.21}\xi_{2t,(l-2)1t} + \varphi`_{2.22}\xi_{2t(l-2),2t} + \cdots + \varphi_{p.21}\xi_{2t,1t(l-p)} + \varphi_{p.22}\xi_{2t,2t(l-p)}$$

$E(\xi_{2t-l}\varepsilon_{1t}) = 0$ (uncorrelated processes)

$$\xi_{2t,2t(l)} = \sum_{s=1}^{m} \varphi_{2s}\ \xi_{2t\,(l),st} + \sum_{k=1}^{p}\sum_{j=1}^{2} \varphi_{k.2j}\, \xi_{2t,jt(l-k)}, \tag{45}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Autocovariance of $Z_{mt}Z_{mt-l}$
$for\ \delta_m = 0, i = m$, Multiplying Equation (43) by Z_{mt-l} and taking the expectations,

$$E(Z_{mt}Z_{mt-l}) = E[Z_{mt-l}\big(\emptyset_{m1}Z_{1t} + \emptyset_{m2}Z_{2t} + \emptyset_{m3}Z_{3t} + \ldots + \emptyset_{m\,(n-1)}Z_{(n-1)t} + \emptyset_{1.m1}Z_{1t-1} + \emptyset_{1.m2}Z_{2t-1} + \cdots + \emptyset_{1.mn}Z_{nt-1} + \emptyset_{2.m1}Z_{1t-2} + \emptyset_{2.m2}Z_{2t-2} + \cdots + \emptyset_{2.mn}Z_{nt-2} + \cdots + \emptyset_{p.m1}Z_{1t-p} + \emptyset_{p.m2}Z_{2t-p} + \cdots + \emptyset_{p.mn}Z_{nt-p} + \varepsilon_{mt}\big)]$$

$$E(Z_{mt}Z_{mt-l}) = \varphi_{m1}E(Z_{mt-l}Z_{1t}) + \varphi_{m2}E(Z_{mt-l}Z_{2t}) + \cdots + \varphi_{m(n-1)}E(Z_{mt-l}Z_{(n-1)t}) + \varphi_{1.m1}E(Z_{mt-l}Z_{1t-1}) + \varphi_{1.m2}E(Z_{mt-l}Z_{2t-1}) + \cdots + \varphi_{1.mn}E(Z_{mt-l}Z_{nt-1}) + \varphi_{2.m1}E(Z_{mt-l}Z_{1t-2}) + \varphi_{2.m2}E(Z_{mt-l}Z_{2t-2}) + \cdots + \varphi_{2.mn}E(Z_{mt-l}Z_{nt-2}) + \cdots + \varphi_{p.m1}E(Z_{mt-l}Z_{1t-p}) + \varphi_{p.m2}E(Z_{mt-l}Z_{2t-p} + \cdots + \varphi_{p.mn}E\big(Z_{mt-l}Z_{nt-p}\big) + E(Z_{mt-l}\varepsilon_{1t})$$

$$\xi_{mt,mt(l)} = \varphi_{m1}\xi_{mt(l),1t} + \varphi_{m2}\xi_{mt(l),3t} + \cdots + \varphi_{m(n-1)}\xi_{mt(l),(n-1)t} + \varphi_{1.m1}\xi_{mt(l-1),1t} + \varphi_{1.m2}\xi_{mt(l-1),2t} + \cdots + \varphi_{1.mn}\xi_{mt(l-1),nt} + \cdots + \varphi`_{2.m1}\xi_{mt,(l-2)1t} + \varphi`_{2.m2}\xi_{mt(l-2),2t} + \cdots + \varphi_{2.mn}\xi_{mt,(l-2)nt} + \cdots + \varphi_{p.m1}\xi_{mt,1t(l-p)} + \varphi_{p.m2}\xi_{mt,2t(l-p)} \ldots + \varphi_{p.mn}\xi_{mt,nt(l-p)},$$

$E(\xi_{mt-l}\varepsilon_{mt}) = 0$ (uncorrelated processes)

$$\xi_{mt,mt(l)} = \sum_{s=1}^{m} \varphi_{ms}\xi_{mt\,(l),st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.mj}\,\xi_{mt,jt(l-k)}, \qquad (46)$$

Autocorrelation of Lower Diagonal MARDL Models
The Autocorrelation of Z_{1t} and Z_{1t-l}

$$\rho_{1t,1t(l)} = \frac{\xi_{1t,1t(l)}}{\xi_{1t,1t}},$$

Where $\xi_{1t,1t}$ is given as $\sum_{s=2}^{m} \varphi_{1s}\,\xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.11}\,\xi_{1t,jt(k)} + \sigma^2_{w_{1t}}, \quad s \neq 1;$ Usoro and Udoh (2023)

Therefore, from Equation (44)

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p} \varphi_{k.11}\,\xi_{1t,kt(l-p)}}{\sum_{s=2}^{m} \varphi_{1s}\,\xi_{1s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.11}\,\xi_{1t,jt(k)} + \sigma^2_{w_{1t}}} & , (s \neq 1),\ l = 1,2,3,\ldots \end{cases} \qquad (47)$$

$\rho_{1t,1t(l)}$ is the autocorrelation of ξ_{1t} *and* $\xi_{1t(l)}$

The Autocorrelation of Z_{2t} and Z_{2t-l}

$$f\,\rho_{2t,2t(l)} = \frac{\xi_{2t,2t(l)}}{\xi_{2t,2t}},$$

Where $\xi_{2t,2t}$ is given as $\sum_{s=1}^{m} \varphi_{2s}\,\xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\,\xi_{2t,jt(k)} + \sigma^2_{w_{2t}},$ $s \neq 2;$ Usoro and Udoh (2023)

Therefore, from Equation (45)

$$\rho_{2t,2t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{s=1}^{m} \varphi_{2s}\ \xi_{2t\,(l),st} + \sum_{k=1}^{p}\sum_{j=1}^{2} \varphi_{k.2j}\,\xi_{2t,jt(l-k)}}{\sum_{s=1}^{m} \varphi_{2s}\,\xi_{2s} + \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\,\xi_{2t,jt(k)} + \sigma^2_{w_{2t}}} & , (s \neq 2),\ l = 1,2,3,\ldots \end{cases} \qquad (48)$$

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$\rho_{2t,2t(l)}$ is the autocorrelation of ξ_{2t} *and* $\xi_{2t(l)}$

$$\vdots \qquad \vdots \qquad \vdots$$

The Autocorrelation of Z_{mt} and Z_{mt-l}

$$\rho_{mt,mt(l)} = \frac{\xi_{mt,mt(l)}}{\xi_{mt,mt}},$$

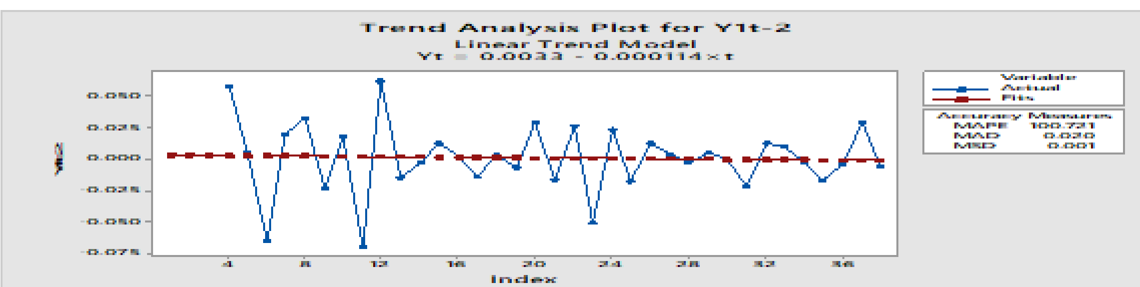
Where $\xi_{mt,mt}$ is given as $\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{ms} + \sum_{k=1}^{p} \sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(k)} + \sigma^2_{w_{mt}}, \quad s \neq m$; Usoro and Udoh (2023)

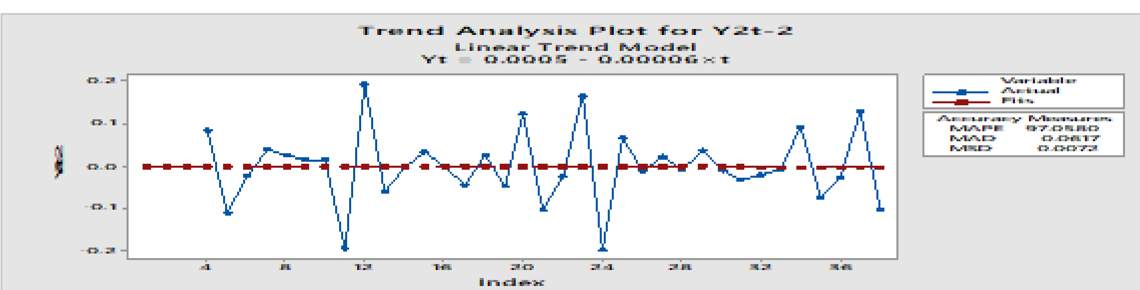
Therefore, from Equation (46)

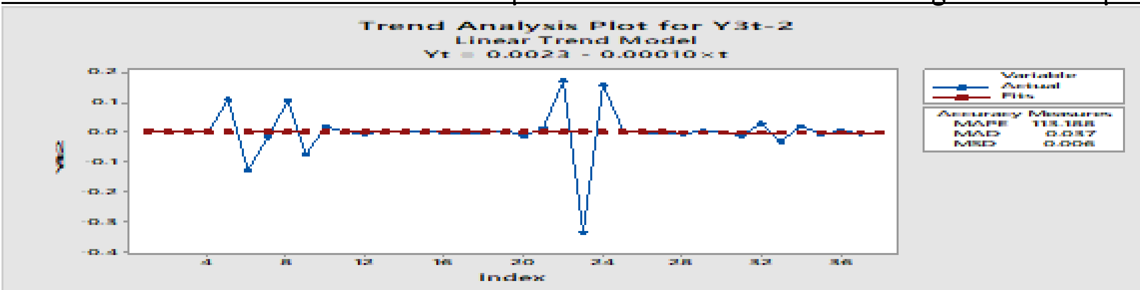
$$\rho_{mt,mt(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{s=1}^{m} \varphi_{ms} \xi_{mt\,(l),st} + \sum_{k=1}^{p} \sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(l-k)}}{\sum_{s=1}^{m-1} \varphi_{ms}\, \xi_{ms} + \sum_{k=1}^{p} \sum_{j=1}^{n} \varphi_{k.mj}\, \xi_{mt,jt(k)} + \sigma^2_{w_{mt}}} & , s \neq m, l = 1,2,3, \ldots \end{cases} \qquad (49)$$

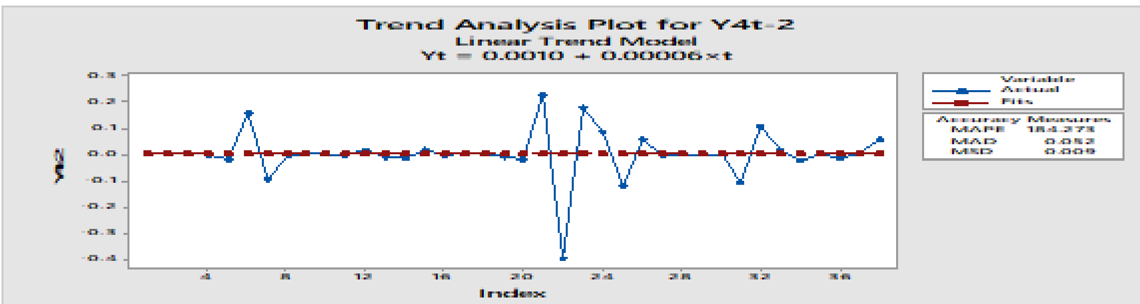
$\rho_{mt,mt(l)}$ is the autocorrelation of ξ_{mt} *and* $\xi_{mt(l)}$

Trend Analysis:









3. Empirical Results

In this section, we examine outcomes derived from the diagonal VAR and MARDL models. These encompass the autocovariance and autocorrelations from both models. The validation of the model was done using data on Nigeria's Gross Domestic Product, Crude Oil Petroleum, Agricultural production, and Telecommunication for the analysis and estimation of model parameters. The data source is the CBN Statistical Bulletin covering the period from 1988 to 2020. The findings are showcased in the tables provided below.

S/N		VAR		MARDL	
		UPPER	**LOWER**	**UPPER**	**LOWER**
1	δ_1	0.0128	0.01940	0.00118	0.00765
2	δ_2	0.00343	0.00338	0.00849	0.00561
3	δ_3	0.02449	0.02423	0.00605	0.00432
4	δ_4	0.04198	0.04229	0.0019	0.0121
5	$\xi_{1t,1t}$	0.000147	0.000119	0.000072	0.000137
6	$\xi_{2t,2t}$	0.069944	0.000196	0.000272	0.0007
7	$\xi_{3t,3t}$	0.026252	0.001000	0.329621	0.000051
8	$\xi_{4t,4t}$	0.001198	0.001650	0.000177	0.0010
9	$\xi_{1t,1t(1)}$	−0.000107	0.000109	0.000001	0.000132
10	$\xi_{1t,1t(2)}$	0.000031	0.000028	0.000007	0.000193
11	$\xi_{2t,2t(1)}$	0.001351	0.000148	0.000111	−0.000079
12	$\xi_{2t,2t(2)}$	-0.000490	−0.000191	−0.000223	−0.000161
13	$\xi_{3t,3t(1)}$	0.000927	0.000273	-0.011400	−0.000020

14	$\xi_{3t,3t(2)}$	−0.000377	−0.000368	−0.000316.	−0.000023
15	$\xi_{4t,4t(1)}$	0.001168	−0.000246	−0.000107	−0.000631
16	$\xi_{4t,4t(2)}$	0.000536	−0.000968	−0.000104	0.000966
17	$\rho_{1t,1t(1)}$	-0.727891	0.915966	0.013888	0.963503
17	$\rho_{1t,1t(2)}$	0.210884	0.235294	0.037222	-0.576642
18	$\rho_{2t,2t(1)}$	0.019315	-0.755102	0.408088	0.219346
19	$\rho_{2t,2t(2)}$	-0.000700	-0.974489	-0.819852	-0.230000
20	$\rho_{3t,3t(1)}$	0.035311	0.273000	-0.034585	-0.392156
21	$\rho_{3t,3t(2)}$	-0.014360	-0.368000	-0.000035	-0.450980
22	$\rho_{4t,4t(1)}$	0.974958	-0.143030	0.914529	0.631000
23	$\rho_{4t,4t(2)}$	0.447412	-0.586666	-0.587570	0.966000

Discussion and Conclusion

The purpose of this work was to determine the autocovariance and the autocorrelations properties of the upper and lower diagonals of VAR and MARDL models. Usoro and Udoh (2023) established the prerequisites for identifying the diagonal VAR and MARDL models and their validity. The performances of the new classes of multivariate lag models were tested using data from certain macroeconomic variables such as Nigeria's Gross Domestic Product (GDP), Crude Oil Petroleum (C/PET), Agriculture (AGRIC), and Telecommunication (TELECOM) used after the first order difference of the logarithm of the series to achieve stationarity. Using the model parameters, the models were estimated, and the Autocovariances and autocorrelation of the processes were derived. According to the findings, the two exhibit both converse and inverse linear relationships. As a result, the negative autocorrelation in the macroeconomic variables implies that the periods of economic expansion were followed by contractions and vice versa. This implies that the two models complement one another when it comes to modelling multivariate lag variables.

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