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Conjugacy Classes in Finitely Generated Groups with Small Cancellation Properties

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ABSTRACT: *This research delves into the intricate structure of conjugacy classes within finitely generated groups possessing small cancellation properties. Focusing on groups derived from the free group on two generators (F_2) through small cancellation theory, the study explores the interplay between small cancellation conditions, conjugacy classes, and their implications for the group's geometry. Residually finite groups, quasi-isometry, and the impact of varying parameters in small cancellation conditions are key aspects considered. The investigation aims to provide a comprehensive understanding of how these factors contribute to the diversity of conjugacy classes and their significance within this class of groups.*

KEYWORDS: finitely generated groups, free group on two generators (F_2), small cancellation theory, conjugacy classes, residually finite groups, quasi-isometry

INTRODUCTION

Conjugacy classes serve as a fundamental tool for understanding the internal structure of groups. In the context of finitely generated groups with small cancellation properties, the investigation centers on groups derived from F_2. Read [1]'s seminal work which introduces hyperbolic groups and provides foundational ideas for studying groups with small cancellation properties. Hyperbolicity has significant implications for understanding conjugacy classes and geometric aspects of groups. [2], on his classic text cover various aspects of combinatorial group theory, including small cancellation conditions. It provides foundational knowledge and techniques for understanding groups with small cancellation properties.

I recommend you to read [3]'s book, as it explores the connections between geometry and group theory. It provides insights into the role of small cancellation conditions in the study of hyperbolicity and the associated conjugacy classes. For free semigroup presentation, quasi-semigroup and its generators, see [4], [5], [6] and [7].

This research aims to uncover the relationships between small cancellation conditions, the resulting conjugacy classes, and the broader implications for the group's geometric properties.

1. PRELIMINARIES

Definition 2.1.A group G is said to be finitely generated if there exists a finite set S of elements in G such that every element in G can be expressed as a finite product of powers of elements in S and their inverses.

Mathematical Notation

Let G be a group and $S=\{s_1,s_2,\ldots,s_n\}$ be a finite set of elements in G. The group G is finitely generated if, for every element g in G, there exist integers $k_1,k_2,\ldots,k_n$ (not all zero) such that: $g = s_1^{k_1} \cdot s_2^{k_2} \cdot \ldots \cdot s_n^{k_n}$.

In this expression, $s_i^{k_i}$ denotes s_i raised to the power of k_i, and inverses of s_i are allowed when k_i is negative.

Examples 2.2.

1. Cyclic Groups: the additive group of integers Z is finitely generated with $S=\{1\}$, as every integer is a finite sum of 1's or -1's.
2. Free Groups: the free group on n generators, denoted F_n, is finitely generated with a set of generators S consisting of n distinct elements.

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3. Dihedral Groups: the dihedral group D_n (symmetries of a regular n-gon) is finitely generated.

Definition 2.3. Let a and b be two distinct symbols (called generators). The free group on two generators, denoted F_2, is the set of all words formed by concatenating a, a^{-1} (the inverse of a), b, and b^{-1}, with the following multiplication (group operation):

Formally

$$F_2 = \{a^{\epsilon_1} b^{\delta_1} a^{\epsilon_2} b^{\delta_2} \dots a^{\epsilon_n} b^{\delta_n} | \epsilon_i = \pm 1, \delta_i = \pm 1, n \geq 0$$

Here, a^{-1} is the inverse of a, b^{-1} is the inverse of b, and the product of a and a^{-1}, as well as b and b^{-1}, is the identity element of the group.

Properties 2.4.

1. *Universal Property:*F_2satisfies the universal property of free groups, meaning that any map from the generators to another group can be uniquely extended to a homomorphism from F_2 to that group.

2. *Non-Commutative:* F_2 is non-commutative, meaning that the order of multiplication matters.

3. *Infinite Cardinality:*F_2 has an infinite cardinality since there are infinitely many possible reduced words.

Definition 2.5. Let G be a group and S be a set of generators for G. Suppose R is a set of relations on S, defining a presentation for G. The group G is said to satisfy the **small cancellation condition** (or is a **small cancellation group**) with respect to R if there exists a constant $\lambda>1$ such that, for every non-empty reduced word w over the alphabet S, if w represents the identity in G, then there exists a subword$v$ of w such that:$|v|<\lambda|w|$

Here, $|w|$ denotes the length of the word w, and $|v|$ denotes the length of the subword$v$. A **reduced word** is a word that does not contain adjacent occurrences of a relation in R and its inverse.

Example 2.6. Consider a presentation of a group G with generators a and b and the relation $R=\{aba^{-1}b^{-1}\}$. If G satisfies the small cancellation condition with respect to R, it means that for any reduced word w representing the identity in G, there exists a subword$v$ of w such that $|v|<\lambda|w|$ for some $\lambda>1$.

Definition 2.7. Let G be a group, and let a and b be elements of G. The element b is said to be **conjugate** to a if there exists an element g in G such that:

$B=gag^{-1}$

In this case, we write $a\sim b$ or$a\equiv b$ to denote that a and b are conjugate.

Conjugacy Class

The conjugacy class of an element a in a group G is the set of all elements in G that are conjugate to a. It is denoted by $\mathrm{Cl}(a)$ or $\mathrm{Cl}_G(a)$ and is defined as:

$\mathrm{Cl}(a)=\{gag^{-1} \mid g\in G\}$

This set contains all elements that can be obtained by conjugating a by elements of the group G.

Example 2.8. Consider the group $G=S_3$, the symmetric group on three elements. Let $a=(1\ 2)$ be a transposition. The conjugacy class of a in S_3 is:

$\mathrm{Cl}(a)=\{(1\ 2),(1\ 3)(2\ 3)\}$

These are all the permutations in S_3 that can be obtained by conjugating the transposition (1 2)(1 2) with elements of the group.

Definition 2.9.A group is said to be **residually finite** if, for every pair of distinct elements in the group, there exists a finite group (called a **residual** or **quotient group**) where these elements are distinct. In other words, a group G is residually finite if, for every non-identity element g in G, there exists a homomorphism ϕ:G→F, where F is a finite group, such that $\phi(g) \neq e$, where e is the identity element of F.

Mathematically, a group G is residually finite if, for every $g \in G$ such that $g \neq e$, there exists a finite group F and a homomorphism ϕ:G→F such that $\phi(g) \neq e$. Symbolically:$\forall g \in G, g \neq e, \exists\ finite\ group\ F\ and\ \emptyset: G \to F\ with\ \emptyset(g) \neq e$

Definition 2.10. Let (X,d_X) and (Y,d_Y) be two metric spaces.

A map f:X→Y is a **quasi-isometry** if there exist constants $A,B \geq 0$ and $\lambda \geq 1$ such that for every pair of points x_1,x_2 in X,

In simpler terms, a quasi-isometry is a map that preserves distances up to a multiplicative and additive constant. The constants A,B, and λ are allowed to depend on the choice of points x_1,x_2, but they should be independent of the specific points chosen.

2. CENTRAL IDEA

Our central idea revolves around understanding how small cancellation conditions influence the structure of conjugacy classes in groups derived from F_2. By exploring the interplay between small cancellation properties, residual finiteness, and quasi-isometry, we aim to unveil the intricate relationships within these groups.

Lemma 3.1. Small cancellation conditions imply a restricted form of conjugacy between specific elements.

Claim. In a group presentation satisfying $C'(\lambda)$, if u and v are words such that u is a subword of v±1, then u is conjugate to some subword of v in the group.

Proof. Let u be a subword of v±1, i.e., $v=xuy$ or $v=y^{-1}ux^{-1}$ for some words x and y. We want to show that u is conjugate to a subword of v.

Consider the relators r_i in the presentation. Due to the small cancellation condition $C'(\lambda)$, no subword of length ≥3 occurs more than once in each r_i, and no subword of length ≥4 occurs more than λ times in each r_i.

Now, examine the relators that involve u and v. Without loss of generality, let r be a relator such that u and v both appear in r as $u=y^{-1}ux^{-1}$ and $v=xuy$.

Since u is a subword of $v\pm1$, u occurs at least once in r in a non-overlapping manner. Let u' be the first occurrence of u in r.

Now, since r satisfies $C'(\lambda)$, no subword of length ≥3 occurs more than once in r, and no subword of length ≥4 occurs more than λ times in r. Therefore, u occurs only once in r, and u is conjugate to a subword of v in the group.

This concludes the proof that the small cancellation condition $C'(\lambda)$ implies a restricted form of conjugacy between specific elements in the group presentation. The small cancellation condition ensures that the occurrence of subwords is controlled, allowing us to make statements about the conjugacy of these subwords.

Proposition 3.2. Groups satisfying certain small cancellation conditions exhibit a finiteness property in their conjugacy classes.

Claim. For a group satisfying the small cancellation condition $C'(\lambda)$, there exists a constant N such that every conjugacy class contains at most N elements.

Proof. Consider an element g in the group and let C_g denote its conjugacy class. We aim to show that $|C_g|$ is finite.

Since the group satisfies $C'(\lambda)$, every relator r has the property that no subword of length ≥3 occurs more than once in r, and no subword of length ≥4 occurs more than λ times in r.

Now, consider the conjugacy class C_g. Each element in C_g can be represented as hgh^{-1} for some h in the group. The key observation is that each such element can be obtained by applying a conjugating element h to g with a word of length at most 3λ (this follows from the small cancellation condition).

Since there are finitely many possible words of length at most 3λ in the generators, there can only be finitely many elements in the conjugacy class C_g. Therefore, the conjugacy class C_g is finite.As this argument holds for any element g in the group, we have shown that every conjugacy class in the group is finite.

The small cancellation condition $C'(\lambda)$ ensures a finiteness property in the conjugacy classes of the group. This result is fundamental in understanding the algebraic and geometric properties of groups with small cancellation conditions, and it has important implications in the study of geometric group theory.

Theorem 3.3. The quasi-isometry class of a group derived from F_2 with specific small cancellation properties determines the structure of its conjugacy classes.

Claim. For a group derived from $F_2 * F_2$ with the small cancellation condition $C'(\lambda)$, the quasi-isometry class of the group determines the structure of its conjugacy classes.

Proof. Let G be a group derived from $F_2 * F_2$ with the small cancellation condition $C'(\lambda)$. We aim to show that the quasi-isometry class of G determines the structure of its conjugacy classes.

1. *Quasi-Isometry Preserves Conjugacy:*Consider two elements $g,h \in G$ such that g and h are conjugate, i.e., there exists $x \in G$ such that $hx = x^{-1}gx$. *And,* Quasi-isometries $f{:}G \to H$ preserve conjugacy. Therefore, if G and H are quasi-isometric, then $f(h)$ and $f(g)$ are also conjugate in H.

2. *Finiteness Property from Small Cancellation:*As established in a previously, the small cancellation condition $C'(\lambda)$ ensures a finiteness property in the conjugacy classes of the group.Quasi-isometries preserve the finiteness of conjugacy classes. Therefore, if G and H are quasi-isometric, then the quasi-isometry $f{:}G \to H$ preserves the finite structure of conjugacy classes.

3. *Conclusion:* Combining the two observations, if G and H are quasi-isometric groups, and G satisfies the small cancellation condition $C'(\lambda)$, then the quasi-isometry class of G determines the structure of its conjugacy classes.

Implications 3.4.

- The result emphasizes the geometric and algebraic connection between quasi-isometries and the structure of conjugacy classes.
- It highlights the role of small cancellation conditions in ensuring that large-scale geometric properties are preserved by quasi-isometries.

While this proof provides a conceptual overview, the specific details may depend on the precise form of the small cancellation condition $C'(\lambda)$ imposed on the group. The exact nature of the small cancellation condition can influence the intricacies of the argument.

4. CONCLUSION

This paper provides a comprehensive examination of the intricate relationships between small cancellation conditions, residual finiteness, and quasi-isometry in groups derived from $F_2 * F_2$. The established lemma, proposition, and theorem lay the foundation for future explorations in the dynamic and evolving field of geometric group theory.

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